

C^r DENSITY OF STABLE ERGODICITY FOR A CLASS OF PARTIALLY HYPERBOLIC SYSTEMS

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ABSTRACT. We show that among a class of skew-product C^r partially hyperbolic volume-preserving diffeomorphisms satisfying some pinching, bunching condition with certain type of dominated splitting in the centre subspace, a C^r dense C^2 open subset contains ergodic diffeomorphisms for $r > 3$. As another application of our techniques, we partially generalised the result in [8] and obtain stable transitivity for action of random rotations on the sphere in arbitrary dimension.

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1. INTRODUCTION

The study of partially hyperbolic systems extends those of uniformly hyperbolic systems, which were developed at length since the sixties. One of the important achievements is accomplished by Anosov, who showed, among other things, that C^2 uniformly hyperbolic volume-preserving diffeomorphisms are always ergodic. Fundamental to Anosov's proof are the existence of stable/unstable manifolds and their absolute continuity. For partially hyperbolic systems, one can also construct stable/unstable manifolds and show their absolute continuity. We refer the readers to [12] for details. It is interested to know to what extend one can recover the results from uniformly hyperbolic systems in the presence of *a priori* non-hyperbolic behaviours. One of the important conjectures is the Stable Ergodicity conjecture, proposed by Pugh-Shub in 1996:

CONJECTURE 1 (Pugh-Shub). *Stable ergodicity is C^r -dense among the C^r partially hyperbolic volume-preserving diffeomorphisms on a compact connected manifold, for any $r > 1$.*

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In their original formulation of the conjecture [16], the term stable ergodicity was referring to C^r -stably ergodicity, i.e. the persistence of ergodicity under C^r small perturbations. More precisely, a diffeomorphism f is called C^r -stable ergodic if there exists $\epsilon > 0$ such that any diffeomorphism g satisfying $d_{C^r}(f, g) < \epsilon$ is ergodic. Later the term stable ergodicity is often referring to the persistence of ergodicity under C^1 small perturbations. In some case one can show the C^r -density of C^1 -stable ergodicity, see for example [1, 6, 19].

This conjecture was supported by a proposed mechanism for ergodicity called "accessibility", which we now come to define.

Definition 1. A partially hyperbolic diffeomorphism $f : X \rightarrow X$ is accessible if any point in X can be reached from any other along an su -path, which is a concatenation of finitely many subpaths, each of which lies entirely in a single leaf of \mathcal{W}^s or a single leaf of \mathcal{W}^u .

A natural weakening of the notion accessibility is the following.

Definition 2. The accessibility class of $p \in X$ is the set of all $q \in M$ that can be reached from p along an su -path.

A partially hyperbolic diffeomorphism $f : X \rightarrow X$ is essentially accessible if every measurable set that is a union of entire accessibility classes has either full or zero volume.

The Stable Ergodicity conjecture was split into two parts using the concept of accessibility. Pugh, Shub conjectured that: 1. Accessibility holds for an open and dense subset of C^r partially hyperbolic diffeomorphisms; 2. A partially hyperbolic C^2 volume preserving diffeomorphism with the essential accessibility property is ergodic.

Significant efforts had been made to study the C^1 version of the conjecture, namely proving ergodicity in a C^1 open and dense subspace in the space of C^1 partially hyperbolic volume-preserving diffeomorphisms. In [9], the authors proved that C^1 -density of C^1 -stable accessibility among partially hyperbolic systems. In [5], the authors showed that essential accessibility plus a mild technical condition called "centre bunching" implies ergodicity.

Finally, the C^1 version of the Stable Ergodicity conjecture was proved in the recent work [2] using a very different approach. They developed a sophisticated generalisation of Hopf's argument relying on a geometric device, the so-called "superblender". It is worth mentioning that their method combining accessibility with some "local" ergodicity mechanism still rely on [9] to deduce stable metric transitivity.

On comparison, there is a paucity of results for the original formulation of the conjecture, namely the C^r density of stable ergodicity. We will give a quick summary of the known results. In [19], the conjecture is proved in general for 1D centre bundles. Recently, A. Avila and M. Viana show the density of C^r -accessibility for a class of skew products with 2D centers [1]. In [6] the authors verified C^r -density of stable ergodicity for group extensions over Anosov diffeomorphisms (for related work, see [7]). For group extensions, one can assume arbitrary centre dimension, but it is clear that the dynamics along the centre directions are isometries and therefore avoided many difficulties that would arise for nonlinear dynamics.

It seems that all the existing approaches for Stable Ergodicity conjecture require proving the density of stable accessibility or essentially accessibility. The problem of proving the C^r -density of accessibility is addressed in many literatures on partially hyperbolic systems (see for example [2], [21]).

In this paper, we prove the C^r density of C^2 stable essential accessibility among a class of skew products satisfying some pinching, centre bunching conditions and certain type of dominated splitting. Combining [4] (or the stronger result [5]), this implies the C^r density of C^2 stable ergodicity among this class. As mentioned above, previous results in this direction either assume the centre dimension is very low (1D or 2D), or assume that there is no non-linearity in the centre direction (One should note that for group extensions, the dynamics along the centre direction are isometries, and any iterations can be described by finitely many real valued functions on the base). Compared to the previous works, ours is the first result that works for any central dimension and unrestricted nonlinearity.

Our proof is divided into two parts. On the one hand, for any C^r skew product map satisfying some pinching, centre bunching conditions, we show that we can produce stable open accessible class under arbitrarily small C^r -perturbations. On the other hand, we show that for a generic skew product satisfying some centre bunching condition, if the centre direction (the fiber tangent space) admits certain type of dominated splitting, then either it has no open accessible class, or it is essentially accessible. The main theorems are as follows.

Theorem 1. *Given an integer $c \geq 2$, $r > 2$, let \mathcal{U} be the set of C^r skew products with central dimension c , satisfying $\frac{c-1}{c}$ -pinching and 1-center bunching conditions. Then there exists a C^r dense C^1 open subset of $\mathcal{U}_0 \subset \mathcal{U}$ such that any diffeomorphism in \mathcal{U}_0 has an open accessible class.*

Here we will define skew product maps in Definition 1, Section 3. The pinching, centre bunching conditions will be given in Section 2. In [14], we proved Theorem 1 and similar results for a more general class of partially hyperbolic systems, though still assuming some pinching condition and certain regularity of the central foliations. For the convenience of the reader, we give a self-contained proof of Theorem 1 in Section 7 using some estimates in Section 4.

Theorem 2. *Given an integer $c \geq 2$, $r > 3$. Let \mathcal{U} be either $\mathcal{U}_1^r(X, Vol; l)$ for some $l \in [1, \frac{c}{2}]$ or $\mathcal{U}_2^r(X, Vol)$ (defined in Section 3) where $X = Y \times N$ and with fiber dimension $\dim N = c$. Then there exists a C^r dense C^2 open subset $\mathcal{U}_0 \subset \mathcal{U}$ such that any diffeomorphism in \mathcal{U}_0 is either essentially accessible or it has no open accessible class.*

Here $\mathcal{U}_1^r(X, Vol; l)$, $\mathcal{U}_2^r(X, Vol)$ are collections of C^r -skew products satisfying some pinching, centre bunching condition with certain type of dominated splitting in the centre subspaces. The precise definition will be given in Definition 11 and 12. As an immediate corollary, we have the following result.

Theorem 3. *Given an integer $c \geq 2$, $r > 3$. Let \mathcal{U} be either $\mathcal{U}_1^r(X, Vol; l)$ for some $l \in [1, \frac{c}{2}]$ or $\mathcal{U}_2^r(X, Vol)$ where $X = Y \times N$ with fiber dimension $\dim N = c$; let \mathcal{V} be the set of C^r skew products on X satisfying $\frac{c-1}{c}$ -pinching condition. Then there exists a C^r dense C^2 open subset of $\mathcal{U} \cap \mathcal{V}$ containing only ergodic diffeomorphisms.*

As an application, we have the following result for linear transforms of the tori.

COROLLARY A. *Given any integer $n, m \geq 2$, and $A \in SL(n, \mathbb{Z})$, $B \in SL(m, \mathbb{Z})$. We denote the linear transform induced by A (resp. B) on torus \mathbb{T}^n (resp. \mathbb{T}^m) as f_A (resp. f_B). Assume the following conditions are satisfied.*

- (1) f_A is $\frac{n}{n+1}$ -pinching;
- (2) f_B is Anosov and $\frac{n-1}{n}$ -pinching.

then for all sufficiently large integer $k \geq 1$, any $r > 3$, there exists a C^1 -neighborhood of the skew-product map $f_B^k \times f_A : \mathbb{T}^m \times \mathbb{T}^n \rightarrow \mathbb{T}^m \times \mathbb{T}^n$ (with base map being f_B^k) in the space of C^r -volume preserving skew products, such that it has a C^r -dense C^2 -open subset containing only ergodic diffeomorphisms.

In [11], the author showed that all pseudo-Anosov linear transform on the \mathbb{T}^n with $2D$ -center are C^r -stable ergodic ($r = 5$ for $n \geq 6$, $r = 22$ for $n = 4$). Compare to this result, our result can show robust ergodicity near linear transform with many zero Lyapunov exponents and even non-ergodic ones. We also relaxed the regularity constraint. But our result fail short of dealing with perturbations without preserving the C^r product structure.

In order to see the difficulty in getting a C^r -density result for accessibility or essentially accessibility, it is helpful to recall the proof of C^1 -density of stable accessibility. In [9], the authors proved C^1 -density of accessibility by combining 1. local accessibility, that is creating accessible class containing some highly non-recurrent disk; 2. proving accessibility modulo a set of well-distributed disks. In the first step, they method of creating local accessible class is based on basic homotopy theory. The main estimate takes the advantage of the fact if one does not have to worry about C^2 norm, one can effectively promote C^0 displacement of the su-paths. This allows one to effectively relate the size of the local accessible class to the size of the perturbation.

For C^r small perturbations, the method in [9] no longer works. Besides, it is unlikely that one would still get good estimates relating the size of the accessible to the room of perturbations, which also makes it useless to using the covering argument to resolve the accessibility problem in the large scale.

Our approach for creating open accessible classes is base on transversality. We will use a topological result borrowed from [3] to show that stable accessible class exist when we have some transversality condition involving the images of sub-manifolds of a map from some parameter space to phase space.

Given a stably open accessible class, it is natural to try to create minimality of the action of holonomy maps by perturbation. It appears that the available results in this direction is unsatisfactory for our purpose. Our main observation is that : for iterations of several conservative maps, we can conclude transitivity from an uniform lower bound of the quality of the stable manifolds for the associated random dynamics (actually our proof shows that any proper closed invariant set has zero measure).

As another application of our method, we generalised the result in [8] and obtain the following.

Theorem 4. *Given $d \in \mathbb{N}^*$, there exists a number k_0 such that for any m , for any set of rotations R_1, \dots, R_m in SO_{d+1} such that R_1, \dots, R_m generate SO_{d+1} there exists a number $\epsilon > 0$ such that if $\{f_\alpha\}$ is a set of volume preserving diffeomorphisms on S^d and $\max_\alpha d_{C^{k_0}}(R_\alpha, f_\alpha) < \epsilon$, then any closed $\{f_\alpha\}$ -invariant set has zero Lebesgue measure.*

In particular, $\{f_\alpha\}$ is transitive and the orbit of almost every point in S^d under the action of $\{f_\alpha\}$ is dense.

This theorem is proved via combining Proposition 8 in Section 6 and the linearization result in [8]. In [8], the authors obtained stable ergodicity of actions of random rotations in even dimensions. For the moment, we still do not know whether we have stable ergodicity in odd dimensions.

Finally, we note that our method rely on the conservation of the holonomy maps, which is conjecture to be a rare phenomenon among volume preserving partially hyperbolic diffeomorphisms (see [20]). Moreover, although our criteria for IFS is C^1 -robust, in order to ensure the C^1 closeness of the holonomy maps one need to impose the C^2 closeness of the partially hyperbolic systems, this is why we are only able to show C^2 -stable ergodicity. To solve the Stable ergodicity completely using accessibility approach one might need to look for a more topological argument.

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2. PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

Given a smooth d -dimensional Riemannian manifold X with a volume form m .

Definition 3. [Anosov, partially hyperbolic and dominated splittings]

A C^1 diffeomorphism $f : X \rightarrow X$ is uniformly hyperbolic or Anosov if there exists a continuous splitting of the tangent bundle into Tf -invariant sub-bundles:

$$TX = E^s \oplus E^u$$

such that for every unit vector $v \in TX$

$$\begin{aligned} \|Df(v)\| &< 1 \text{ if } v \in E^s, \\ \|Df(v)\| &> 1 \text{ if } v \in E^u. \end{aligned}$$

A C^1 diffeomorphism $f : X \rightarrow X$ is partially hyperbolic if the following conditions hold. There is a nontrivial continuous splitting of the tangent bundle $TX = E^s \oplus E^c \oplus E^u$, that is invariant under Df . Furthermore, there exists $\bar{\chi}^u, \bar{\chi}^s > 0$ and $\bar{\chi}^c, \hat{\chi}^c \in \mathbb{R}$ such that

$$(2.1) \quad -\bar{\chi}^s < \bar{\chi}^c \leq \hat{\chi}^c < \bar{\chi}^u$$

and we have

$$(2.2) \quad \|Df(v)\| < e^{-\bar{\chi}^s} \|v\|, \forall v \in E^s \setminus \{0\}$$

$$(2.3) \quad e^{\bar{\chi}^c} \|v\| < \|Df(v)\| < e^{\hat{\chi}^c} \|v\|, \forall v \in E^c \setminus \{0\}$$

$$(2.4) \quad e^{\bar{\chi}^u} \|v\| < \|Df(v)\|, \forall v \in E^u \setminus \{0\}$$

We denote the set of all the C^r (resp. C^r volume preserving) partially hyperbolic diffeomorphisms by $\mathcal{PH}^r(X)$ (resp. $\mathcal{PH}^r(X, m)$).

A C^1 diffeomorphism $f : X \rightarrow X$ has a non-trivial uniformly dominated splitting if the following conditions hold. There is a nontrivial continuous splitting of the tangent bundle $TX = \oplus_{i=1}^k E_i$, $k \geq 2$ that is invariant under Df . Furthermore, there exists a collection of constants $\{\tilde{\chi}_i, \hat{\chi}_i\}_{1 \leq i \leq k-1}$ such that for any $1 \leq i \leq k-1$, we have

$$(2.5) \quad \tilde{\chi}_i < \hat{\chi}_i, \forall 1 \leq i \leq k-1$$

$$(2.6) \quad \|Df(x, v)\| < e^{\tilde{\chi}_i} \|v\|, \forall v \in E_i(x) \setminus \{0\}$$

$$(2.7) \quad e^{\hat{\chi}_i} \|u\| < \|Df(x, u)\|, \forall u \in E_{i+1}(x) \setminus \{0\}$$

For any partially hyperbolic diffeomorphism $f : X \rightarrow X$, through each point $x \in X$ (including all the C^r Anosov maps), we have a well-defined unstable manifold $\mathcal{W}_f^u(x)$ and stable manifold $\mathcal{W}_f^s(x)$. Moreover, X is foliated by \mathcal{W}_f^u , \mathcal{W}_f^s , and $\mathcal{W}_f^c(x)$. $\mathcal{W}_f^u(x)$, $\mathcal{W}_f^s(x)$ are C^r -manifolds, though the transverse regularity is often merely Hölder.

Given any two sufficiently close submanifolds $\mathcal{D}_1, \mathcal{D}_2$ transversal to the \mathcal{W}_f^u foliation, for some subset of \mathcal{D}_1 denoted by \mathcal{D}_0 , for each $x \in \mathcal{D}_0$, the local unstable manifold $\mathcal{W}_f^u(x)$ intersect \mathcal{D}_2 at a unique point, denoted by $H_{f, \mathcal{D}_1, \mathcal{D}_2}^u(x)$. Then the map $x \mapsto H_{f, \mathcal{D}_1, \mathcal{D}_2}^u(x)$ is a well-defined continuous map from a subset of \mathcal{D}_0 to \mathcal{D}_2 . We call $H_{f, \mathcal{D}_1, \mathcal{D}_2}^u$ the unstable holonomy map from \mathcal{D}_1 to \mathcal{D}_2 . We define stable holonomy maps in a similar way.

Definition 4. [Dynamically coherence] A partially hyperbolic diffeomorphism $f : X \rightarrow X$ is dynamically coherent if E^{cs} and E^{cu} are integrable to foliations \mathcal{W}^{cs} and \mathcal{W}^{cu} respectively. In this case, E^c is integrable to the central foliation \mathcal{W}^c for which $\mathcal{W}^c(x) = \mathcal{W}^{cs}(x) \cap \mathcal{W}^{cu}(x)$ for all $x \in X$.

It is an open question that whether dynamical coherence is a C^1 open condition. It is known that when E^c is C^1 and integrable, dynamical coherence is C^1 robust. Since we will only be focus on skew products, dynamical coherence comes in the definition, so we will not be needing results in this direction.

Given a partially hyperbolic diffeomorphism $f : X \rightarrow X$. Assume that f is dynamically coherent, and the central leaves are compact. For any two compact leaves $\mathcal{C}_1, \mathcal{C}_2$ that are contained in a central unstable leaf \mathcal{W}^{cu} , we have a well-defined holonomy map $H_{f, \mathcal{C}_1, \mathcal{C}_2}^u$ from \mathcal{C}_1 to \mathcal{C}_2 . Moreover $H_{f, \mathcal{C}_1, \mathcal{C}_2}^u$ is defined everywhere on \mathcal{C}_1 . Similarly, the stable holonomy map $H_{f, \mathcal{C}_1, \mathcal{C}_2}^s : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is defined when $\mathcal{C}_1, \mathcal{C}_2$ belong to the same central stable leaf.

Definition 5. [Pinching] A partially hyperbolic system $f \in \mathcal{PH}^r(X)$ is called θ -pinching if the following holds. There exist constants $\hat{\chi}^s, \hat{\chi}^u > 0$ such that

$$\begin{aligned} e^{-\hat{\chi}^s} \|v\| &< \|Df(v)\|, \forall v \in TX \\ \|Df(v)\| &< e^{\hat{\chi}^u} \|v\|, \forall v \in TX \end{aligned}$$

and

$$\begin{aligned} -\tilde{\chi}^s + \theta \hat{\chi}^u &< \tilde{\chi}^c \\ \tilde{\chi}^u - \theta \hat{\chi}^s &> \hat{\chi}^c \end{aligned}$$

Here $\tilde{\chi}^s, \tilde{\chi}^u, \tilde{\chi}^c, \hat{\chi}^c$ are constants satisfying (2.1) to (2.4).

Our interest in partially hyperbolic diffeomorphisms satisfying pinching conditions comes from the following theorem in [17].

Theorem 5. *Suppose that $f : X \rightarrow X$ is a θ -pinching C^r partially hyperbolic system, then the local unstable and stable holonomy maps are uniformly θ -Holder.*

We also introduce the following notion of pinching for diffeomorphisms with a dominated splitting.

Definition 6. [Pinching for dominated splittings] A diffeomorphism $f : X \rightarrow X$ having a non-trivial uniformly dominated splitting $TX = \oplus_{i=1}^k E_i, k \geq 2$ is called θ -pinching if the following holds. There exist constants $(\bar{\chi}_i, \hat{\chi}_i)_{1 \leq i \leq k-1}$ satisfying (2.5), (2.6), (2.7) and $\hat{\chi}^u, \hat{\chi}^s$ such that

$$(2.8) \quad e^{-\hat{\chi}^s} \|v\| < \|Df(v)\|, \forall v \in TX$$

$$(2.9) \quad \|Df(v)\| < e^{\hat{\chi}^u} \|v\|, \forall v \in TX$$

and

$$(2.10) \quad \max(\hat{\chi}^u, \hat{\chi}^s)\theta < \min(\hat{\chi}_{k-1} - \bar{\chi}_{k-1}, \hat{\chi}_1 - \bar{\chi}_1)$$

It is clear from the definition that any diffeomorphism $f \in \mathcal{DS}^r(X)$ is θ -pinching for some $\theta > 0$. Our interest in Definition 6 comes from the following proposition.

PROPOSITION 1. *Let $r \geq 2, \theta \in (0, 1]$. Let $f : X \rightarrow X$ be a C^r diffeomorphism having a θ -pinching uniformly dominated splitting $TM = \oplus_{i=1}^k E_i, k \geq 2$ with constants $(\bar{\chi}_i, \hat{\chi}_i)_{1 \leq i \leq k-1}, \hat{\chi}^s, \hat{\chi}^u$ satisfying (2.5) to (2.10), then $E_1(x), E_k(x)$ are θ -Holder functions of x .*

Proof. Denote $d_1 = \dim(E_1)$. Denote the d_1 -subspace Grassmannian bundle over X by Gr_1 and the projection $p_1 : Gr_1 \rightarrow X$. There is a natural smooth structure on Gr_1 . We denote the lift of $f : X \rightarrow X$ to Gr_1 by $F_1 : Gr_1 \rightarrow Gr_1$. Since f is C^r , F_1 is C^{r-1} . By the Df -invariance of the splitting, we see that the image of the map $x \mapsto (x, E_1(x))$, denoted by Σ_1 , is a F_1 -invariant section. By (2.6), we can take an open neighborhood of Σ_1 , denote by U_1 such that

$$F_1^{-1}(U_1) \subset U_1$$

and the F_1^{-1} restricted to U_1 is fibre contraction (see [12]) with contraction rate stronger than $e^{-\hat{\chi}_1 + \bar{\chi}_1}$. More precisely, for each $x \in X$, any $v_1, v_2 \in U_1 \cap p^{-1}(\{x\})$ we have

$$d(F_1^{-1}(v_1), F_1^{-1}(v_2)) < e^{-\hat{\chi}_1 + \bar{\chi}_1} d(v_1, v_2)$$

By (2.9), the strongest contraction of f^{-1} on the base is weaker than $e^{-\hat{\chi}^u}$. That is, for any two points $x_1, x_2 \in X$ that are sufficiently close, we have

$$d(f^{-1}(x_1), f^{-1}(x_2)) \geq e^{-\hat{\chi}^u} d(x_1, x_2)$$

Then by (2.10) and the Holder section theorem in [12], we conclude that $F_1^{-1}|_{U_1}$ has a unique invariant section, which is θ -Holder. Thus $E_1(x)$ is a θ -Holder function of x . Similarly, we can show that $E_k(x)$ is also a θ -function of x . This completes the proof. \square

We have the following immediate corollary of Proposition 1.

COROLLARY B. *Under the conditions of Proposition 1, there exists $\epsilon > 0$ such that $E_1(x), E_k(x)$ are $(\theta + \epsilon)$ -Holder function of x .*

Proof. By (2.10), there exists $\epsilon > 0$ such that (2.10) holds after replacing θ by $\theta + \epsilon$. This shows that f is $(\theta + \epsilon)$ -pinching. We finish the proof by applying Proposition 1. \square

It is helpful to consider the induced dynamics on varies Grassmannian bundles of a manifold which we now define.

Definition 7. For any integer $l \in [1, d-1]$, we denote $Gr(X, l)$ the Grassmannian bundle of X and $p_l : Gr(X, l) \rightarrow X$ the canonical projection. For each $x \in X$, $p_l^{-1}(x)$ is identified with $Gr(T_x X, l)$, the Grassmannian of l dimensional subspaces of $T_x X$, in a natural way. We denote an element in $Gr(X, l)$ by (x, E) , here E is a l dimensional subspace of $T_x X$.

Given a C^r diffeomorphisms $f : X \rightarrow X$, there is a canonical C^{r-1} diffeomorphism $\mathbb{G}(f) : Gr(X, l) \rightarrow Gr(X, l)$ associated with f , define as follows.

$$\mathbb{G}(f)(x, E) = (f(x), Df(x, E))$$

We have equality $p_l \mathbb{G}(f) = f p_l$.

Definition 8. [l -center bunching] A partially hyperbolic system $f : X \rightarrow X$ is called l -center bunching if the following holds.

$$-\tilde{\chi}^s - \tilde{\chi}^c + k\hat{\chi}^c < 0$$

and

$$-\tilde{\chi}^u + \tilde{\chi}^c - k\hat{\chi}^c < 0$$

for all $1 \leq k \leq l$. Here $\tilde{\chi}^s, \tilde{\chi}^u, \tilde{\chi}^c, \hat{\chi}^c$ are constants satisfying (2.1) to (2.4).

We denote the set of C^r l -center bunching (resp. volume preserving) partially hyperbolic diffeomorphisms by $\mathcal{PH}_{bun}^r(X, l)$ (resp. $\mathcal{PH}_{bun}^r(X, m, l)$).

The following theorem is essentially proved in [17].

Theorem 6. Given $r \geq 2$. For $1 \leq l \leq r$, suppose that $f : X \rightarrow X$ is a C^r l -center bunching, dynamically coherent partially hyperbolic diffeomorphism, then the local unstable and stable holonomy maps between centre leaves are C^l .

The following is also contained in the proof of main theorem in [17] and will be used in several places.

Theorem 7. Given number $r > 2$. Suppose that $f : X \rightarrow X$ is a C^r 1 -center bunching, dynamically coherent partially hyperbolic diffeomorphism, then the local unstable and stable holonomy maps between centre leaves are $C^{1+\beta}$ for some $\beta > 0$.

3. SKEW PRODUCTS AND EXAMPLES

The class of partially hyperbolic systems we will be studying is a class of "non-linear" skew products in contrast to the skew products considered in [6] which was referring to compact group extensions. This class of partially hyperbolic systems is also under the name "smooth cocycles", for example in [1] and the reference therein.

NOTATION 1. Let Y be a d_0 -dimensional compact Riemannian manifold with a volume form μ . Let f be a volume preserving Anosov diffeomorphism on Y . Let $c \in \mathbb{N}, c \geq 2$ and let N be a c -dimensional compact Riemannian manifold with a

volume form m . We say a C^r partially hyperbolic systems $F : Y \times N \rightarrow Y \times N$ is a C^r skew product if

$$F(y, z) = (f(y), g(y, z))$$

with centre directions at $(y, z) \in Y \times N$ given by $E^c(y, z) = \{0\} \times T_z N$. We say that f is the *base map*. Moreover, we call F a C^r volume preserving skew product if for each $y \in Y$, $g(y, \cdot) \in \text{Diff}^r(N, m)$.

It is direct to check that a C^r volume preserving skew product F preserves measure $\mu \times m$. Hence $F \in \mathcal{PH}^r(Y \times N, \mu \times m)$.

For each integer $r \geq s \geq 1$, we will define the C^s -topology on the space of C^r skew products to be the topology induced by the C^s -distance on the space of C^r -partially hyperbolic systems. Thus if for two C^r -skew products F, \tilde{F} defined as $F(y, z) = (f(y), g(y, z))$ and $\tilde{F}(y, z) = (\tilde{f}(y), \tilde{g}(y, z))$, we have $d_{C^r}(F, \tilde{F}) < \epsilon$, then we have

- (1) $d_{C^r}(f, \tilde{f}) < \epsilon$;
- (2) $\sup_{y \in Y} d_{C^r}(g(y, \cdot), \tilde{g}(y, \cdot)) < \epsilon$.

For any skew product $F : Y \times N \rightarrow Y \times N$, all compact leaves of F are of the form $N_y := \{y\} \times N$ for some $y \in Y$. Moreover, for any $\sigma > 0$, we denote

$$B(N_y, \sigma) = B(y, \sigma) \times N$$

We will define two classes of diffeomorphisms as follows. Using these diffeomorphisms, we can construct transitive iterated function systems as shown in Lemma 9,10. Later on we will use these lemmata to construct C^2 -stably ergodic partially hyperbolic diffeomorphisms by C^r perturbations.

Definition 9. Given a c -dimensional Riemannian manifold X with a volume form m . For integers $r \geq 2$, $l \in [1, \frac{c}{2}]$, we denote $\mathcal{DS}_1^r(X, m; l)$ the set of C^r volume preserving diffeomorphisms F such that there exists an uniformly dominated splitting

$$T_x X = E_1(x) \oplus E_2(x) \oplus E_3(x)$$

such that $\dim E_1 = \dim E_3 = l$.

Definition 10. Given a c -dimensional Riemannian manifold X with a volume form m . Let $\mathcal{DS}_2^r(X, m)$ denote the set of C^r volume preserving diffeomorphisms that satisfy that following conditons. For each $f \in \mathcal{DS}_2^r(X, m)$, there exists an uniformly dominated splitting

$$TX = E_1 \oplus E_2$$

with $\dim(E_1) \leq \dim(E_2)$. Let constants $\tilde{\chi}_1, \hat{\chi}_1 \in \mathbb{R}$ be given by Definition 3 related to the above splitting, there exist constants $\chi^u, \chi^s \in \mathbb{R}$ that satisfy

$$e^{-\chi^s} \|v\| \leq \|Df(x, v)\| \leq e^{\chi^u} \|v\|, \forall x \in X, v \in T_x X$$

and

$$(3.1) \quad \tilde{\chi}_1 - 2\hat{\chi}_1 + \chi^u < 0$$

We now define two classes of partially hyperbolic skew products. The fiber maps of these two classes of skew products resemble those diffeomorphisms defined in Definition 9 and 10.

Definition 11. Let $(Y \times N, \mu \times m)$ be defined in Notation 1. For integers $r \geq 2$, $l \in [1, \frac{c}{2}]$, we denote $\mathcal{U}_1^r(Y \times N, \mu \times m; l)$ the set of skew products $F : Y \times N \rightarrow Y \times N$ such that the centre space admits an uniformly dominated splitting

$$E^c = \{0\} \times TN = E_1 \oplus E_2 \oplus E_3$$

such that $\dim E_1 = \dim E_3 = l$. Moreover, we require the following. Let $\bar{\chi}^s, \bar{\chi}^u, \bar{\chi}^c, \hat{\chi}^c$ be the constants for the partially hyperbolic splitting and let $\{\bar{\chi}_i, \hat{\chi}_i\}_{1 \leq i \leq 2}$ be the constants for the above dominated splitting i.e. for $i = 1, 2$

$$\sup_{v \in E_i \setminus \{0\}} \frac{\|DF(v)\|}{\|v\|} < e^{\bar{\chi}_i} < e^{\hat{\chi}_i} < \inf_{u \in E_{i+1} \setminus \{0\}} \frac{\|DF(u)\|}{\|u\|}$$

we require that

$$(3.2) \quad \max(-\bar{\chi}^c, \hat{\chi}^c) \frac{c}{c+1} < \min_{1 \leq i \leq 2} (\hat{\chi}_i - \bar{\chi}_i)$$

$$(3.3) \quad -\bar{\chi}^{s,u} + \hat{\chi}^c - \bar{\chi}^c + \max(\hat{\chi}^c, 0) + \max(-\bar{\chi}^c, 0) < 0$$

Definition 12. Let $(Y \times N, \mu \times m)$ be defined in Notation 1. For integers $r \geq 2$, we denote $\mathcal{U}_2^r(Y \times N, \mu \times m)$ the set of skew products $F : Y \times N \rightarrow Y \times N$ such that the centre space admits an uniformly dominated splitting

$$E^c = \{0\} \times TN = E_1 \oplus E_2$$

such that $\dim E_1 \leq \dim E_2$. Moreover we require the following. Let $\bar{\chi}^s, \bar{\chi}^u, \bar{\chi}^c, \hat{\chi}^c$ be the constant for the partially hyperbolic splitting and let $\bar{\chi}_1, \hat{\chi}_1$ be the constants for the above dominated splitting i.e.

$$e^{\bar{\chi}^c} < \sup_{v \in E_1 \setminus \{0\}} \frac{\|DF(v)\|}{\|v\|} < e^{\bar{\chi}_1} < e^{\hat{\chi}_1} < \inf_{u \in E_2 \setminus \{0\}} \frac{\|DF(u)\|}{\|u\|} < e^{\hat{\chi}^c}$$

Moreover, we require that

$$(3.4) \quad \max(-\bar{\chi}^c, \hat{\chi}^c) \frac{c}{c+1} < \hat{\chi}_1 - \bar{\chi}_1$$

$$(3.5) \quad \hat{\chi}^c + \bar{\chi}_1 - 2\hat{\chi}_1 < 0$$

$$(3.6) \quad -\bar{\chi}^{s,u} + \hat{\chi}^c - \bar{\chi}^c + \max(\hat{\chi}^c, 0) + \max(-\bar{\chi}^c, 0) < 0$$

REMARK 1. It is clear from the definition that $\mathcal{U}_1^r(Y \times N, \mu \times m; l)$, $l \in [1, \frac{c}{2}]$ and $\mathcal{U}_2^r(Y \times N, \mu \times m)$ are C^1 open sets in the space of C^r skew products.

EXAMPLE 2. Given any $A \in SL(n, \mathbb{Z})$, $n \geq 2$, we denote the affine action induced A on \mathbb{T}^n by $f_A : \mathbb{T}^n \rightarrow \mathbb{T}^n$. We denote the eigenvalue of A by $\lambda_1, \dots, \lambda_n$ sorted by their modulus i.e. $|\lambda_1| \leq |\lambda_2| \leq \dots \leq |\lambda_n|$. We assume that $|\lambda_i|$ are not all equal to 1. Let $b(A)$ be the largest integer in $[1, n]$ such that $|\lambda_{b(A)}| < 1$. By symmetry, it is easy to see that we have $|\lambda_{b(A)+1}| = \dots = |\lambda_{n-b(A)}| = 1$ and $|\lambda_{n-b(A)+1}| > 1$. We denote $\bar{\chi}(A) = \log |\lambda_{n-b(A)+1}|$ and $\hat{\chi}(A) = \log |\lambda_n|$.

Now let $B \in SL(2m, \mathbb{Z})$ induce an Anosov action on \mathbb{T}^{2m} i.e. $b(B) = m$. Let $A \in SL(n, \mathbb{Z})$, $n \geq 2$, whose eigenvalues are not all of modulus 1. Denote $F = f_B \times f_A : \mathbb{T}^{2m} \times \mathbb{T}^n \rightarrow \mathbb{T}^{2m} \times \mathbb{T}^n$ the skew product map with base dynamics being $f_B : \mathbb{T}^{2m} \rightarrow \mathbb{T}^{2m}$. Assume the following conditions hold.

$$(3.7) \quad \bar{\chi}(B) > 4\hat{\chi}(A)$$

$$(3.8) \quad \hat{\chi}(A) \frac{n}{n+1} < \bar{\chi}(A)$$

$$(3.9) \quad \bar{\chi}(B) - \frac{n-1}{n}\hat{\chi}(B) > \hat{\chi}(A)$$

Then for each integer $r \geq 3$, there exist a C^1 neighbourhood of F in the space of C^r –skew products, denoted by \mathcal{U} , such that a C^r dense C^2 open subset of \mathcal{U} contains only ergodic diffeomorphisms.

It is easy to see that (3.9) implies that $\bar{\chi}(B) > n\hat{\chi}(A)$. Thus for (3.7), (3.8) and (3.9) to hold, it suffices to check

$$(3.10) \quad \hat{\chi}(A) \frac{n}{n+1} < \bar{\chi}(A) \text{ and } \bar{\chi}(B) > \max(n, 4)\hat{\chi}(A)$$

In the case where $m = 1$ and A has exactly two eigenvalue of modulus different from 1, then we have $\bar{\chi}(B) = \hat{\chi}(B)$ and $\bar{\chi}(A) = \hat{\chi}(A)$. As a consequence, (3.7), (3.8) and (3.9) reduce to

$$\hat{\chi}(B) > \max(n, 4)\hat{\chi}(A)$$

4. RANDOM PERTURBATIONS

In Section 4 and 5, we will establish some estimates for the perturbation of the holonomy maps of a skew product map.

Given an integer $r \geq 2$. Let X be a compact Riemannian manifold with a volume form Vol and $F \in \mathcal{P}H^r(X, Vol)$. Let $\bar{\chi}^c, \hat{\chi}^c, \bar{\chi}^s, \bar{\chi}^u$ be the constants in Definition 3 and inequalities (2.1) to (2.4) in Definition 3 are satisfied with f replaced by F . We define constant $\xi > 0$ as

$$\xi = \min(\bar{\chi}^c + \bar{\chi}^s, \bar{\chi}^u - \hat{\chi}^c)$$

We will repeatedly use the following definition.

Definition 13. A C^r deformation is defined by the following data. Let $I > 0$ be an integer, \mathcal{U} be an open neighbourhood of the origin in \mathbb{R}^I . Let $\hat{F} : \mathcal{U} \times X \rightarrow X$ be a C^r map such that

- (1) $\hat{F}(0, \cdot) = F$;
- (2) For each $b \in \mathcal{U}$, $\hat{F}(b, \cdot) \in \mathcal{P}H^r(X)$.

We call \hat{F} a smooth deformation of F with I –parameters. We associate with \hat{F} the following map $T : \mathcal{U} \times X \rightarrow \mathcal{U} \times X$ by

$$(4.1) \quad T(b, x) = (b, \hat{F}(b, x))$$

Moreover, if for each $b \in \mathcal{U}$, we have $\hat{F}(b, \cdot) \in \mathcal{P}H^r(X, Vol)$, then we say that \hat{F} is a volume preserving C^r deformation.

It is also convenient to have the following definition.

Definition 14. Given an integer $I > 0$, a smooth map $V : \mathbb{R}^I \times X \rightarrow TX$ is called an infinitesimal C^r deformation of F with I –parameters if

- (1) $V(0, \cdot) \equiv 0$;
- (2) For each $B \in \mathbb{R}^I$, $V(B, \cdot)$ is a C^r vector field on X ;
- (3) For each $x \in X$, $B \mapsto V(B, x)$ is a linear map from \mathbb{R}^I to $T_x X$.

For a sufficiently small $\epsilon > 0$, we can define a C^r deformation of F with I -parameters \hat{F} associated to V using the following formula.

$$\hat{F}(b, x) = \Phi_{V(b, \cdot)}(F(x), 1), \forall (b, x) \in U \times X$$

where $U = B(0, \epsilon) \subset \mathbb{R}^I$ and $\Phi_{V(b, \cdot)} : X \times \mathbb{R} \rightarrow X$ denotes the flow generated by vector field $V(b, \cdot)$.

In this case, we say that \hat{F} is generated by V and V is the generator of \hat{F} . Moreover, if for each $B \in \mathbb{R}^I$, $V(B, \cdot)$ is divergence free, we say that V is a volume preserving infinitesimal C^r deformation. A C^r deformation generated by a volume preserving infinitesimal C^r deformation is volume preserving.

For each $(b, x) \in U \times X$, we have a T -invariant splitting

$$T_b U \oplus T_x X = E_T^u(b, x) \oplus E_T^c(b, x) \oplus E_T^s(b, x)$$

where

$$\begin{aligned} E_T^{u,s}(b, x) &= \{0\} \oplus E_{\hat{F}(b, \cdot)}^{u,s}(x) \\ E_T^c(b, x) &= T_b U \oplus E_{\hat{F}(b, \cdot)}^c(x) \end{aligned}$$

We will use the inclusions $E_{\hat{F}(b, \cdot)}^*(x) \rightarrow \{0\} \oplus E_{\hat{F}(b, \cdot)}^*(x) \subset T_b U \times T_x X$ for $* = s, u, c$ tacitly.

For any $(b, x) \in U \times X$ and $v \in T_b U \times T_x X$, we denote $\pi_X(v)$ the component of v in $T_x X$ and we denote $\pi_*(v)$ the component of v in E_F^* for $* = u, s, c$ and in $T_b U$ for $* = b$.

CAUTION 1. Under our notations, for $v \in T_b U \oplus T_x X$ the component of v in the centre subspace of T is not $\pi_c(v)$ but rather $\pi_c(v) + \pi_b(v)$.

NOTATION 2. Let Y, N be given by Definition 1. From now on till the end of Section 5, we assume that F is a volume preserving skew product and denote

$$\begin{aligned} X &= Y \times N \\ \text{Vol} &= \mu \times m \end{aligned}$$

Recall that for each $y \in Y$, $N_y = \{y\} \times N$ is a central leaf. For $y_1, y_2 \in Y$ contained in an unstable leaf for f (resp. stable leaf for f), we use abbreviations H_{F, y_1, y_2}^u (resp. H_{F, y_1, y_2}^s) to denote $H_{F, N_{y_1}, N_{y_2}}^u$ (resp. $H_{F, N_{y_1}, N_{y_2}}^s$).

We denote by $\pi_Y : X \rightarrow Y$ (resp. $\pi_N : X \rightarrow N$) the canonical projection to Y (resp. N).

From now on we will only be considering C^r deformation of F preserving the central foliation. More precisely, let $\hat{F} : U \times X \rightarrow X$ be a C^r deformation of F , we assume that :

$$(4.2) \quad \text{For each } b \in U, \text{ each } x \in X, \text{ we have } \hat{F}(b, x) \in \mathcal{W}_F^c(F(x))$$

We make a parallel assumption for infinitesimal C^r deformation V as follows

$$(4.3) \quad \text{For each } B \in \mathbb{R}^I, \text{ each } x \in X, \text{ we have } V(B, x) \in E_F^c(x)$$

REMARK 2. The relation between (4.2) and (4.3) is clear. Denote V an infinitesimal C^r deformation of F satisfying (4.3), and denote \hat{F} a C^r deformation generated by V , then \hat{F} satisfy (4.2).

For any C^r deformation \hat{F} satisfying (4.2), the following is clear :

$$E_{\hat{F}(b,\cdot)}^*(x) = E_F^*(x) \text{ and } E_T^*(b,x) = T_b U \oplus E_F^*(x) \text{ for } * = c, cs, cu;$$

In the following, we will use the inclusion $TU \subset E_T^c$ tacitly. For example, in the expression $DT((0,x), B)$ where $B \in T_0 U \simeq \mathbb{R}^l$, we mean that B is identified with an element in $T_0 U \times \{0\} \subset E_T^c(0,x)$ under the natural inclusion.

PROPOSITION 2. *Given $r \geq 2$. Assume that \hat{F} is a C^r deformation of F with l -parameters satisfying (4.2). There exists an open neighborhood of the origin contained in \mathbb{R}^l , denoted by U_0 , such that T restricted to $U_0 \times X$ is a partially hyperbolic systems and dynamically coherent with centre leaves of the form $U_0 \times \mathcal{C}$ where \mathcal{C} is some central leaf of F . Moreover for integer $r \geq l \geq 1$ and $F \in \mathcal{PH}_{\text{bun}}^r(X, l)$, after possibly reducing the size of U_0 , we can assume that the s, u -holonomy maps between centre leaves of T are C^l.*

Proof. By (4.2), the distribution $TU \oplus E^c$ is T -invariant. By the cone field criteria, after possibly reducing the size of U_0 , we can ensure that for all $b \in U_0$, $E_{\hat{F}(b,\cdot)}^u$ is close to E_F^u and along which the expansion rate of $\hat{F}(b, \cdot)$ is close to that of F , and the similar thing for $E_{\hat{F}(b,\cdot)}^s$. Thus we get a continuous splitting satisfying the hypothesis of partially hyperbolic system, with centre subspaces given by $E_T^c = TU \oplus E_F^c$. It is direct to check that E_T^c is integrable to $U_0 \times \mathcal{C}$ where \mathcal{C} is some central leaf of F . This proves the first statement.

For the second statement, it is clear that l -center bunching condition is an C¹-open condition. Then after possibly reducing the size of U_0 , and replacing T by its iterations, we can verify the l -center bunching condition. The smoothness of s, u -holonomies follows from Theorem 6. \square

In the following, we will replace U by U_0 satisfying the conclusion of Proposition 2. When the exponents for the partially hyperbolic splitting of F are given, we will always assume that U is chosen to be small enough so that for any $b \in U$, the same set of exponents works for the splitting of the perturbed map $\hat{F}(b, \cdot)$.

NOTATION 3. Let $\hat{F} : U \times X \rightarrow X$ be a C^r deformation of F , and let \mathcal{C} be any central leaf for F , we denote $\tilde{\mathcal{C}} = U \times \mathcal{C}$. It is direct to see that $\tilde{\mathcal{C}}$ is a central leaf for T . We say $\tilde{\mathcal{C}}$ is the lift of \mathcal{C} for T . Moreover, let $\mathcal{C}, \mathcal{C}'$ be any central leaves for F contained in a central unstable leaf for F , the lift $\tilde{\mathcal{C}}, \tilde{\mathcal{C}}'$ are contained in a central unstable leaf for T . We denote $H_{T, \tilde{\mathcal{C}}, \tilde{\mathcal{C}}'}^u$ the unstable holonomy map between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}'$. Similarly, we define stable holonomy map $H_{T, \tilde{\mathcal{C}}, \tilde{\mathcal{C}}'}^s$ for $\mathcal{C}, \mathcal{C}'$ contained in a central stable leaf for F . For $y_1, y_2 \in Y$ contained in a unstable leaf for f (resp. stable leaf for f), we use abbreviations H_{T, y_1, y_2}^u (resp. H_{T, y_1, y_2}^s) to denote $H_{T, \tilde{N}_{y_1}, \tilde{N}_{y_2}}^u$ (resp. $H_{T, \tilde{N}_{y_1}, \tilde{N}_{y_2}}^s$).

REMARK 3. *Most of the propositions in Section 4,5 can be extended to the case where the ambient manifold is C⁰-foliated by compact central leaves using some of the ideas in [18]. However, our argument linking the transversality to the transitivity exploits the fact that the holonomy maps for this skew-products preserve a smooth measure. This condition is expected to be easily destroyed by a generic volume preserving C^r-perturbation that do not preserve the central foliations ([20]). This is the reason why we focus on skew products with a C^r-central foliation and the diffeomorphisms that preserve it. For symplectomorphisms, the holonomy maps preserve the smooth measure induced by the symplectic structure. In this context, our results can be strengthened. We will not pursue this avenue here.*

Now we will give some estimates related to the holonomy maps. In this section, all the implicit constants (in symbols O, \lesssim , etc.) depend only F .

PROPOSITION 3 (*A priori estimates*). *For integer $r \geq 2$, let F be a C^r 1-center bunching, dynamically coherent partially hyperbolic diffeomorphisms. Let $\hat{F} : U \times X \rightarrow X$ be a C^r deformation of F generated by an infinitesimal C^r deformation V . Then for any central leaves \tilde{C}, \tilde{C}' contained in a local central unstable leaf (resp. local central stable leaf), we have*

$$\begin{aligned} \|DH_{T, \tilde{C}, \tilde{C}'}^u\|_{C^0} &\lesssim \max(\text{Lip}(V), 1) d_{W_T^u}(\tilde{C}, \tilde{C}') \\ (\text{resp. } \|DH_{T, \tilde{C}, \tilde{C}'}^s\|_{C^0} &\lesssim \max(\text{Lip}(V), 1) d_{W_T^s}(\tilde{C}, \tilde{C}')) \end{aligned}$$

Here the implicit constant may depend on F , but not on \hat{F} .

Proof. This is essentially proved in [17]. \square

Some of the estimates will depend on the support of a deformation or an infinitesimal deformation, which we now define.

Definition 15. *For an infinitesimal C^r deformation of F with I -parameters $V : \mathbb{R}^I \times X \rightarrow TX$, we define*

$$\text{supp}_X(V) = \{x \in X \mid \exists B \in \mathbb{R}^I \text{ such that } V(B, x) \neq 0\}$$

For a C^r deformation of F with I -parameters $\hat{F} : U \times X \rightarrow X$, we define

$$\text{supp}_X(\hat{F}) = \{x \in X \mid \exists b \in U \text{ such that } \hat{F}(b, x) \neq x\}$$

It is clear from Definition 14, 15 that for any infinitesimal C^r deformation V , let \hat{F} be a C^r deformation of F generated by V , we have

$$\text{supp}_X(\hat{F}) \subset \text{supp}_X(V)$$

We will make some definitions related to an open set Q .

Definition 16. *Given an open set $Q \subset X$, $C > 0$. Denote $\Delta = \text{diam}(\pi_Y(Q))$. An infinitesimal C^r deformation $V : \mathbb{R}^I \times X \rightarrow TX$ is adapted to (Q, C) if*

$$(4.4) \quad \text{supp}_X(V) \subset Q$$

and

$$(4.5) \quad \text{Lip}(V) \leq C\Delta^{-1}$$

Given any $F \in \text{Diff}^r(X)$, we define

$$R(Q) = \inf\{n > 0 \mid F^{-n}(Q) \cap Q \neq \emptyset \text{ or } F^n(Q) \cap Q \neq \emptyset\}$$

PROPOSITION 4. *For any open set $Q \subset X$, any $C > 0$, any integer $n \geq 1$ such that $R(Q) > n$, let V be an infinitesimal C^r deformation of F that is adapted to (Q, C) , let $x, y \in Q$ and $B \in T_0U$, we have the following :*

- (1) *If $x \notin \text{supp}_X(V)$ and x, y lie in a local unstable leaf of F that is contained in Q , then*

$$\|\pi_c(DH_{T, \pi_Y(x), \pi_Y(y)}^u((0, x), B)) - V(B, y)\| \lesssim Ce^{-n\xi} \|B\|$$

- (2) *If x, y lie in a local stable leaf of F that is contained in Q , then*

$$\|\pi_c(DH_{T, \pi_Y(x), \pi_Y(y)}^s((0, x), B))\| \lesssim Ce^{-n\xi} \|B\|$$

Proof. By the invariance of s, u, c -foliations, we have

$$H_{T, \pi_Y(x), \pi_Y(y)}^* = T^n H_{T, f^{-n}(\pi_Y(x)), f^{-n}(\pi_Y(y))}^* T^{-n}$$

for all $n \in \mathbb{Z}$ and $* = s, u$.

By the definition of T (4.1), for any $(b, x) \in U \times X$, we have

$$DT((b, x); T_x X) = T_x X$$

and

$$\pi_b(DT((b, x); B)) = B \text{ for all } B \in T_b U$$

Since for each $(b, x) \in U \times X$, $\mathcal{W}_T^{s,u}(b, x) \subset \{b\} \times X$, it is easy to see that for any $x, y \in \mathcal{W}_F^s$ (resp. $x, y \in \mathcal{W}_F^u$), for any $B \in T_0 U$, we have

$$(4.6) \quad \pi_b(DH_{T, \pi_Y(x), \pi_Y(y)}^s((0, x), B)) = B$$

$$(4.7) \quad (\text{resp. } \pi_b(DH_{T, \pi_Y(x), \pi_Y(y)}^u((0, x), B)) = B)$$

Moreover, by the F -invariance of E_F^* for $* = s, u, c$ and Definition 13, for any $x \in X$ we have $DT^{-1}((0, x), E_F^*) = E_F^*$ for $* = s, u, c$.

Assume that the conditions in (1) are satisfied. Since $R(Q) > n$, when restricted to $U \times T^{-m}(Q)$ for all $2 \leq m \leq n$ we have

$$DT = Id \times DF$$

In particular, for any $1 \leq m \leq n-1$, any $z \in T^{-m}(Q)$, any $B \in T_0 U$, we have

$$DT^{-1}((0, z), B) = B$$

Moreover, since any $x \notin \text{supp}_X(V)$, we have

$$DT^{-1}((0, x), B) = B$$

As a result, for any x, y in (1), we have that

$$\begin{aligned} & \pi_c(DH_{T, \pi_Y(x), \pi_Y(y)}^u((0, x), B)) \\ &= \pi_c(DT^n(T^{-n}(0, y), B + \pi_c(DH_{T, f^{-n}(\pi_Y(x)), f^{-n}(\pi_Y(y))}^u DT^{-n}((0, x), B)))) \\ (4.8) \quad &= \pi_c(DT^n(T^{-n}(0, y), B + \pi_c(DH_{T, f^{-n}(\pi_Y(x)), f^{-n}(\pi_Y(y))}^u(T^{-n}(0, x), B)))) \end{aligned}$$

We claim that

$$\|\pi_c(DH_{T, f^{-n}(\pi_Y(x)), f^{-n}(\pi_Y(y))}^u(T^{-n}(0, x), B))\| \lesssim Ce^{-n\tilde{\chi}^u} \|B\|$$

Indeed, since x, y lie in a local F -unstable leaf that is contained in Q , we know that

$$d(T^{-n}(0, x), T^{-n}(0, y)) \leq e^{-n\tilde{\chi}^u} d(x, y) \lesssim e^{-n\tilde{\chi}^u} \Delta$$

Then the claim follows from Proposition 3 and (4.5).

By (4.8) and the fact that $\|DT^n|_{E_F^c}\| \lesssim e^{n\tilde{\chi}^c}$, we have

$$\begin{aligned} & \pi_c(DH_{T, (0, x), (0, y)}^u((0, x), B)) \\ &= \pi_c(DT^n(T^{-n}(0, y), \pi_c(DH_{T, f^{-n}(\pi_Y(x)), f^{-n}(\pi_Y(y))}^u(T^{-n}(0, x), B)))) \\ & \quad + \pi_c(DT(T^{-1}(0, y), B)) \\ &= V(B, y) + O(Ce^{n(\tilde{\chi}^c - \tilde{\chi}^u)} \|B\|) \\ &= V(B, y) + O(Ce^{-n\tilde{\zeta}} \|B\|) \end{aligned}$$

Similar argument shows that under the condition in (2), we have

$$\begin{aligned} \pi_c(DH_{T,\pi_Y(x),\pi_Y(y)}^s((0,x),B)) &= O(Ce^{n(-\bar{\chi}^c-\bar{\chi}^s)}\|B\|) \\ &= O(Ce^{-n\bar{\xi}}\|B\|) \end{aligned}$$

□

As a consequence of the above lemmata, we can estimate the derivatives of the map corresponding to holonomy loops. For our purpose, we will be concentrated on the simplest type of holonomy loops described as follows.

Definition 17. Let F be a dynamically coherent partially hyperbolic system with compact central leaves. Given a central leaf \mathcal{C} , a triple $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ denoted by γ is called a 4-legged su -loop at \mathcal{C} if the following holds : $\{\mathcal{C}_i\}_{1 \leq i \leq 3}$ are compact central leaves and $\mathcal{C}_1 \in \mathcal{W}_F^u(\mathcal{C})$, $\mathcal{C}_2 \in \mathcal{W}_F^s(\mathcal{C}_1)$, $\mathcal{C}_3 \in \mathcal{W}_F^u(\mathcal{C}_2)$ and $\mathcal{C} \in \mathcal{W}_F^s(\mathcal{C}_3)$.

To each 4-legged su -loop γ , we assign a homeomorphism of \mathcal{C} , denoted by $H_{F,\gamma}$, defined as follows

$$H_{F,\gamma} = H_{F,\mathcal{C}_3,\mathcal{C}}^s H_{F,\mathcal{C}_2,\mathcal{C}_3}^u H_{F,\mathcal{C}_1,\mathcal{C}_2}^s H_{F,\mathcal{C},\mathcal{C}_1}^u$$

Let $\gamma = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ be a su -loop at \mathcal{C} , we denote $\tilde{\gamma} = (\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2, \tilde{\mathcal{C}}_3)$. It is clear that $\tilde{\gamma}$ is a su -loop at $\tilde{\mathcal{C}}$. We call $\tilde{\gamma}$ the lift of γ for T .

PROPOSITION 5. Given integer $r \geq 2$ and let $F \in \mathcal{PH}_{bun}^r(X, 1)$ be a dynamically coherent partially hyperbolic system with compact central leaves, a compact central leaf \mathcal{C} , an open neighbourhood of \mathcal{C} denoted by Q and $C > 0$. Let $\gamma = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ be a 4-legged su -loop at \mathcal{C} such that the part of stable/unstable leaves connecting $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are contained in Q . Let $\hat{F} : U \times X \rightarrow X$ be a C^r deformation of F with I -parameters with generator V that is adapted to (Q, \mathcal{C}) such that $\mathcal{C}, \mathcal{C}_2, \mathcal{C}_3$ are disjoint from $\text{supp}_X(V)$. Then after possibly replacing U by a smaller domain containing 0, we have $H_{T,\tilde{\gamma}} \in \text{Diff}^1(\tilde{\mathcal{C}})$ and for any $x \in \mathcal{C}$, any $B \in \mathbb{R}^I$ we have

$$\begin{aligned} &\|\pi_c(DH_{T,\tilde{\gamma}}((0,x),B)) - H_{F,\mathcal{C}_3,\mathcal{C}}^s H_{F,\mathcal{C}_2,\mathcal{C}_3}^u \\ &\cdot H_{F,\mathcal{C}_1,\mathcal{C}_2}^s (H_{F,\mathcal{C},\mathcal{C}_1}^u(x), V(B, H_{F,\mathcal{C},\mathcal{C}_1}^u(x)))\| \lesssim Ce^{-R(Q)\bar{\xi}}\|B\| \end{aligned}$$

Here $\tilde{\gamma}$ is the lift of γ for T .

Proof. For $i = 1, 2, 3$, denote $\tilde{\mathcal{C}}_i$ the lift of \mathcal{C}_i for T according to Notation 3. Then

$$H_{T,\tilde{\gamma}} = H_{T,\tilde{\mathcal{C}}_3,\tilde{\mathcal{C}}}^s H_{T,\tilde{\mathcal{C}}_2,\tilde{\mathcal{C}}_3}^u H_{T,\tilde{\mathcal{C}}_1,\tilde{\mathcal{C}}_2}^s H_{T,\tilde{\mathcal{C}},\tilde{\mathcal{C}}_1}^u$$

Since F is 1-bunching, by Proposition 2, $H_{T,\tilde{\mathcal{C}},\tilde{\mathcal{C}}_1}^u, H_{T,\tilde{\mathcal{C}}_1,\tilde{\mathcal{C}}_2}^s, H_{T,\tilde{\mathcal{C}}_2,\tilde{\mathcal{C}}_3}^u$ and $H_{T,\tilde{\mathcal{C}}_3,\tilde{\mathcal{C}}}^s$ are C^1 after possibly reducing the size of U . Thus for any $x \in \mathcal{C}$, $B \in T_0 U$, we have

$$\begin{aligned} (4.9) \quad &DH_{T,\tilde{\gamma}}((0,x),B) \\ &= D(H_{T,\tilde{\mathcal{C}},\tilde{\mathcal{C}}}^s H_{T,\tilde{\mathcal{C}}_2,\tilde{\mathcal{C}}_3}^u H_{T,\tilde{\mathcal{C}}_1,\tilde{\mathcal{C}}_2}^s H_{T,\tilde{\mathcal{C}},\tilde{\mathcal{C}}_1}^u)((0,x),B) \end{aligned}$$

In order to estimate (4.9), we denote

$$\begin{aligned}
I_1(B) &= DH_{T, \mathcal{C}, \mathcal{C}_1}^u((0, x), B) \\
x_1 &= H_{F, \mathcal{C}, \mathcal{C}_1}^u(x) \\
I_2(B) &= DH_{T, \mathcal{C}_1, \mathcal{C}_2}^s((0, x_1), I_1(B)) \\
x_2 &= H_{F, \mathcal{C}_1, \mathcal{C}_2}^s(x_1) \\
I_3(B) &= DH_{T, \mathcal{C}_2, \mathcal{C}_3}^u((0, x_2), I_2(B)) \\
x_3 &= H_{F, \mathcal{C}_2, \mathcal{C}_3}^u(x_2) \\
I_4(B) &= DH_{T, \mathcal{C}_3, \mathcal{C}}^s((0, x_3), I_3(B)) \\
x_4 &= H_{F, \mathcal{C}_3, \mathcal{C}}^s(x_3)
\end{aligned}$$

It is direct to verify that

$$DH_{T, \tilde{\gamma}}((0, x), B) = I_4(B)$$

For the brevity of the notations, we let $\mathcal{C}_4 = \mathcal{C}$. It's clear that $I_i(B) \in E_T^c$ and $x_i \in \mathcal{C}_i$ for each $i = 1, \dots, 4$.

We denote for each $i = 1, \dots, 4$ that

$$I_i^c(B) = \pi_c(I_i(B))$$

Then by (4.6), (4.7) we have

$$I_i(B) = I_i^c(B) + B$$

Thus we have

$$(4.10) \quad I_{i+1}^c(B) = H_{F, \mathcal{C}_i, \mathcal{C}_{i+1}}^*(x_i, I_i^c(B)) + \pi_c(H_{T, \mathcal{C}_i, \mathcal{C}_{i+1}}^*((0, x_i), B))$$

Here $*$ = s when $i = 1, 3$ and $*$ = u when $i = 2$.

By hypothesis $R(Q) > 1$, $\mathcal{C}, \mathcal{C}_2, \mathcal{C}_3$ are disjoint from $\text{supp}_X(V)$ and $F^{-1}(\text{supp}_X(V))$. By Proposition 4, for $i = 1, 2, 3$, take $*$ = s when $i = 1, 3$ and $*$ = u when $i = 2$, we have

$$\|\pi_c(H_{T, \mathcal{C}_i, \mathcal{C}_{i+1}}^*((0, x_i), B))\| \lesssim Ce^{-R(Q)\xi} \|B\|$$

while

$$\|\pi_c(H_{T, \mathcal{C}, \mathcal{C}_1}^u((0, x), B)) - V(x_1, B)\| \lesssim Ce^{-R(Q)\xi} \|B\|$$

Then by (4.10) and Proposition 3, we got

$$I_4^c(B) = H_{F, \mathcal{C}_3, \mathcal{C}}^s H_{F, \mathcal{C}_2, \mathcal{C}_3}^u H_{F, \mathcal{C}_1, \mathcal{C}_2}^s(x_1, V(B, x_1)) + O(Ce^{-R(Q)\xi} \|B\|)$$

□

5. DIMENSION GAP

Using the estimates of the derivatives of holonomy maps, we will estimate the measure of the parameters in a C^r deformation that corresponding to unlikely co-incidences related to sets of large positive codimensions. The main result of this section is Proposition 6. First we need to give lower bounds for certain determinants.

Given $r > 2$, a 1-center bunching C^r skew product $F : X \rightarrow X$, we have the following lemma. In the following, all the implicit constants depends only on F .

LEMMA 1. For any $L \in \mathbb{N}$, $C, \kappa > 0$, there exists $R_0 = R_0(L, C, \kappa) > 0$ such that the following is true. Let \mathcal{C} be a central leaf, let Q be an open neighbourhood of \mathcal{C} , let $\gamma_1, \dots, \gamma_L$ be L su-loops at \mathcal{C} , denoted by $\gamma_i = (\mathcal{C}_{i,1}, \mathcal{C}_{i,2}, \mathcal{C}_{i,3})$ for $1 \leq i \leq L$, such that the parts of stable/unstable leaves connecting $\mathcal{C}, \mathcal{C}_{i,1}, \mathcal{C}_{i,2}, \mathcal{C}_{i,3}$ are contained in Q . Let $V : \mathbb{R}^I \times X \rightarrow TX$ be an infinitesimal C^r deformation adapted to (Q, \mathcal{C}) . Assume that :

(1) $R(Q) > R_0$

(2) Denote $B = (B_\alpha)_{1 \leq \alpha \leq I} \in \mathbb{R}^I$. For any $x_1, \dots, x_L \in \mathcal{C}$, there exist $\{\alpha_{l,k}\}_{\substack{1 \leq l \leq L \\ 1 \leq k \leq c}} \subset [I]$ such that

$$(5.1) \quad D_{B_{\alpha_{j,1}}, \dots, B_{\alpha_{j,c}}} (V(B, H_{F, \mathcal{C}, \mathcal{C}_{i,1}}^u(x_i))) = 0$$

for all $1 \leq i \neq j \leq L$, and

$$(5.2) \quad |\det(D_{B_{\alpha_{i,1}}, \dots, B_{\alpha_{i,c}}} (V(B, H_{F, \mathcal{C}, \mathcal{C}_{i,1}}^u(x_i))))| > 2\kappa$$

for all $1 \leq i \leq L$;

(3) $\mathcal{C}, \mathcal{C}_{i,2}, \mathcal{C}_{i,3}$, $1 \leq i \leq L$ are disjoint from $\text{supp}_X(V)$.

Let \hat{F} be a C^r deformation of F with I -parameters generated by V . Then for any $x_1, \dots, x_L \in \mathcal{C}$, there exists a subspace $H \subset \mathbb{R}^I$ such that $\dim(H) = Lc$ and

$$\det(H \ni B \mapsto (\pi_c(DH_{T, \tilde{\gamma}_i}((0, x_i), B)))_{i=1, \dots, L} \in \prod_{i=1}^L T_{H_{F, \gamma_i}(x_i)} \mathcal{C}) \geq \kappa^L$$

Proof. Given $x_1, \dots, x_L \in \mathcal{C}$, let $H = \oplus_{i=1}^L \oplus_{k=1}^c \mathbb{R} \partial_{B_{\alpha_{i,k}}}$, where $\{\alpha_{i,k}\}_{\substack{1 \leq i \leq L \\ 1 \leq k \leq c}}$ are given by (2). Define $\psi : H \rightarrow \prod_{i=1}^L T_{H_{F, \gamma_i}(x_i)} \mathcal{C}$ by

$$\psi(B) = (\pi_c(DH_{T, \tilde{\gamma}_i}((0, x_i), B)))_{i=1, \dots, L}$$

For each $i = 1, \dots, L$, let $\{v_{i,1}, \dots, v_{i,c}\}$ be a set of orthonormal basis of $T_{H_{F, \gamma_i}(x_i)} \mathcal{C}$, let $\{u_{i,1}, \dots, u_{i,c}\}$ be a set of orthonormal basis of $T_{H_{F, \mathcal{C}, \mathcal{C}_{i,1}}(x_i)} \mathcal{C}_{i,1}$. Let the $Lc \times Lc$ -matrix G represent the map ψ under the basis $\{\partial_{B_{\alpha_{i,k}}}\}_{\substack{1 \leq i \leq L \\ 1 \leq k \leq c}}$ and $\{v_{i,k}\}_{\substack{1 \leq i \leq L \\ 1 \leq k \leq c}}$, then

$$\det(\psi) = \det(G)$$

Let the $c \times c$ -submatrix of G , denoted by $G_{i,j}$, represent the map $B \mapsto \pi_c(DH_{T, \tilde{\gamma}_j}^u((0, x_j), B))$ restricted to $\oplus_{k=1}^c \mathbb{R} \partial_{B_{\alpha_{i,k}}}$ under the basis $\{\partial_{B_{\alpha_{i,k}}}\}_{1 \leq k \leq c}$ and $\{v_{j,k}\}_{1 \leq k \leq c}$;

Let the $c \times c$ -matrix, denoted by $K_{i,j}$, represent the map $B \mapsto \pi_c(DH_{T, \tilde{\mathcal{C}}, \tilde{\mathcal{C}}_{j,1}}^u((0, x_j), B))$ restricted to $\oplus_{k=1}^c \mathbb{R} \partial_{B_{\alpha_{i,k}}}$ under the basis $\{\partial_{B_{\alpha_{i,k}}}\}_{1 \leq k \leq c}$ and $\{u_{j,k}\}_{1 \leq k \leq c}$;

Let the $c \times c$ -matrix, denoted by $E_{i,j}$, represent the map $B \mapsto V(B, H_{F, \mathcal{C}, \mathcal{C}_{j,1}}(x_j))$ restricted to $\oplus_{k=1}^c \mathbb{R} \partial_{B_{\alpha_{i,k}}}$ under the basis $\{\partial_{B_{\alpha_{i,k}}}\}_{1 \leq k \leq c}$ and $\{u_{j,k}\}_{1 \leq k \leq c}$;

Let the $c \times c$ -matrix, denoted by A_j , represent the map $T_{H_{F, \mathcal{C}, \mathcal{C}_{j,1}}(x_j)} \mathcal{C} \ni u \mapsto D(H_{F, \mathcal{C}_{j,3}, \mathcal{C}}^u H_{F, \mathcal{C}_{j,2}, \mathcal{C}_{j,3}}^u H_{F, \mathcal{C}_{j,1}, \mathcal{C}_{j,2}}^s)(H_{F, \mathcal{C}, \mathcal{C}_{j,1}}^u(x_j), u) \in T_{H_{F, \gamma_j}(x_j)} \mathcal{C}$ under the basis $\{u_{j,k}\}_{1 \leq k \leq c}$ and $\{v_{j,k}\}_{1 \leq k \leq c}$.

It is direct to see that we have

$$(5.3) \quad G_{i,j} = A_j K_{i,j}$$

By Proposition 5 and (1), we have

$$\|A_j E_{i,j} - G_{i,j}\| \lesssim C e^{-R_0 \xi}$$

Since A_j represents the derivatives of concatenations of local holonomy maps, by Lemma 3 and the hypothesis that V is (Q, C) -adapted, we have

$$(5.4) \quad \|A_j\| = O(\text{Lip}(V)\text{diam}(Q)) = O(C)$$

and by (5.1), we have

$$(5.5) \quad \|E_{i,j}\| = 0$$

for $1 \leq i \neq j \leq L$, and by (5.2) we have

$$(5.6) \quad \det(E_{i,i}) > 2\kappa$$

for all $1 \leq i \leq L$. By (5.3), (5.4), (5.5), for $1 \leq i \neq j \leq L$ we have,

$$\|G_{i,j}\| \lesssim Ce^{-R_0\zeta}$$

By (5.3), (5.4), (5.6), for all $1 \leq i \leq L$ we have

$$\det(G_{i,i}) \gtrsim 2\kappa - O(Ce^{-R_0\zeta})$$

when R_0 is sufficiently large depending only on L, C, κ . Thus we have

$$\det(G) \gtrsim \kappa^L$$

when R_0 is sufficiently large depending only on L, C, κ .

This concludes the proof. \square

Now we have come to the main proposition of this section.

PROPOSITION 6. *Let $L > 0$ be an integer, $\mathcal{C} \subset X$ be a centre leaf, $\gamma_1, \dots, \gamma_L$ be L su-leaves at \mathcal{C} . Denote $\gamma_i = (C_{i,1}, C_{i,2}, C_{i,3})$ for $1 \leq i \leq L$. Let \hat{F} be a C^r deformation of F with I -parameters. Moreover there exists $\kappa > 0$ such that for any $x \in \mathcal{C}$, there exists a subspace $H \subset \mathbb{R}^I$ such that $\dim(H) = Lc$, and*

$$(5.7) \quad \det(H \ni B \mapsto (\pi_c(DH_{T, \tilde{\gamma}_i}((0, x), B)))_{i=1, \dots, L}) \in \prod_{i=1}^L T_{H_{F, \gamma_i}(x)} \mathcal{C} \geq \kappa$$

Then for any $\Sigma \subset \mathcal{C}^L$ such that

$$(5.8) \quad HD(\Sigma) < c(L-1)$$

for any $\epsilon > 0$, there exists $b \in U$ such that

- (1) $\|b\| < \epsilon$;
- (2) Denote $F' = \hat{F}(b, \cdot)$. For any $x \in \mathcal{C}$, we have

$$(H_{F', \gamma_i}(x))_{1 \leq i \leq L} \notin \Sigma$$

Proof. For each $x \in \mathcal{C}$, we define map

$$\begin{aligned} \Psi_x : U &\rightarrow \mathcal{C}^L \\ b &\mapsto (\pi_c(H_{T, \tilde{\gamma}_i}(b, x)))_{1 \leq i \leq L} = (H_{\hat{F}(b, \cdot), \gamma_i}(x))_{1 \leq i \leq L} \end{aligned}$$

For any $B \in T_0U$,

$$D\Psi_x((0, x), B) = (\pi_c(DH_{T, \tilde{\gamma}_i}((0, x), B)))_{1 \leq i \leq L} \in \prod_{i=1}^L T_{H_{F, \gamma_i}(x)} \mathcal{C}$$

Then by (5.7) there exists $H \subset T_0U$ such that $\dim(H) = Lc$ and $\det(D\Psi_x(0, x)|_H) > \kappa$. By continuity, there exists $\zeta > 0$, such that for each $b \in U$ and $|b| < \zeta$, there

exists $H' \subset T_b U$ such that $\dim(H') = Lc$ and $\det(D\Psi_x(b, x)|_{H'}) > \kappa$. By Theorem 7 and the equicontinuity of the Holder functions, we see that ζ can be chosen to be independent of x . Hence for any $x \in \mathcal{C}$, any $b \in U$, $|b| < \zeta$, there exists $H \subset T_b U$ such that $\dim(H) = Lc$ and $\det(D\Psi_x(b, x)|_H) > \kappa$. We denote $U_0 = U \cap B_U(0, \zeta)$.

For any $\delta > 0$, we denote

$$\Sigma_\delta = \{z \in \mathcal{C}^L | d(z, \Sigma) < \delta\}$$

It is a classical fact that there exists constant $C_4 > 0$ such that

$$\text{Vol}_{\mathcal{C}}^L(\Sigma_\delta) \leq C_4 \delta^{Lc - HD(\Sigma)}$$

Here $\text{Vol}_{\mathcal{C}}$ denotes the volume form induced by the Riemannian metric on \mathcal{C} .

We choose some sufficiently small $\delta > 0$, some small constant $\beta > 0$, choose a $\delta^{1+\beta}$ net in \mathcal{C} , denoted by \mathcal{N} . Since $\dim(\mathcal{C}) = c$, we have $\#\mathcal{N} \lesssim \delta^{-(1+\beta)c}$. For each $x \in \mathcal{N}$ we have

$$\text{Vol}((\Psi_x|_{U_0})^{-1}\Sigma_\delta) \lesssim \kappa^{-1} \text{Vol}_{\mathcal{C}}^L(\Sigma_\delta)$$

Consider

$$U_1 = \bigcup_{x \in \mathcal{N}} (\Psi_x|_{U_0})^{-1}(\Sigma_\delta)$$

We have

$$\begin{aligned} \text{Vol}(U_1) &\leq \#\mathcal{N} \times \kappa^{-1} \text{Vol}_L(\Sigma_\delta) \\ &\lesssim \kappa^{-1} \delta^{cL - c(1+\beta) - HD(\Sigma)} \end{aligned}$$

By (5.8), when $\beta > 0$ is sufficiently small, the exponent of δ , i.e. $cL - c(1 + \beta) - HD(\Sigma)$, is positive. Then $\text{Vol}(U_1)$ tends to 0 as δ tends to 0. Thus for any $\epsilon > 0$, when δ is sufficiently small, we have $U_1 \subsetneq U_0$ and there exists $b \in U_0 \setminus U_1$ such that $d_{C^r}(\hat{F}(b, \cdot), F) < \epsilon$. Define $F' = \hat{F}(b, \cdot)$. We claim that F' satisfies the conclusion of Proposition 6.

Indeed, for each $x \in \mathcal{C}$, there exists $y \in \mathcal{N}$ such that

$$(5.9) \quad d(x, y) \leq \delta^{1+\beta}$$

Since $b \in U_0 \setminus U_1$, we have

$$d((H_{F', \gamma_i}(y))_{1 \leq i \leq L}, \Sigma) > \delta$$

By (5.9) and Proposition 3, for each $1 \leq i \leq L$, we have

$$d(H_{F', \gamma_i}(x), H_{F', \gamma_i}(y)) \lesssim \delta^{1+\beta}$$

Thus

$$d((H_{F', \gamma_i}(x))_{1 \leq i \leq L}, (H_{F', \gamma_i}(y))_{1 \leq i \leq L}) \lesssim L\delta^{1+\beta}$$

This shows that

$$d((H_{F', \gamma_i}(y))_{1 \leq i \leq L}, \Sigma) > \delta - O(L\delta^{1+\beta}) > 0$$

when δ is sufficiently small. In particular, we have

$$(H_{F', \gamma_i}(x))_{1 \leq i \leq L} \notin \Sigma$$

This concludes the proof. \square

6. DICHOTOMY : ESSENTIALLY ACCESSIBLE VERSUS THE NON-EXISTENCE OF OPEN ACCESSIBLE CLASSES

6.1. A criteria for the transitivity of iterated function systems. In this section, we will recall some basic notions and results in the study of the dynamics of multiple diffeomorphisms which are relevant to our results.

Let M be a compact d -dimensional Riemannian manifold with a smooth volume form m . Given an integer $k \geq 2$ and k diffeomorphisms $f_1, \dots, f_k \in \text{Diff}^1(M)$, we are interested in the dynamics of the iterations of these k diffeomorphisms.

Definition 18. For any $k \geq 2$, $f_1, \dots, f_k \in \text{Diff}_m^1(M)$, we call $\{f_i\}_{1 \leq i \leq k}$ an IFS (i.e. iterated function system).

Similar to the study of the dynamics of a single diffeomorphism, we have analogous notions of ergodicity and transitivity.

An IFS $\{f_i\}_{1 \leq i \leq k}$ is transitive if there exists $x \in M$ such that for any open set $U \subset M$, there exists a finite words $\omega = \omega_1 \omega_2 \dots \omega_l$ in $1, \dots, k$ such that $f_{\omega_l} \dots f_{\omega_1}(x) \in U$.

Definition 19. Given an integer $k \geq 2$, an IFS $\{f_i\}_{1 \leq i \leq k}$. We denote $\Omega = \{1, \dots, k\}^{\mathbb{Z}}$ and denote \mathbb{P} the Bernoulli probability measure on Ω . Then (Ω, \mathbb{P}) is the probability space of two-sided infinite sequences of independent random variables uniformly distributed on $\{1, \dots, k\}$. For any $n \geq 1$, $\omega = (\dots \omega_{-1} \omega_0 \omega_1 \dots) \in \Omega$, we denote

$$f_{\omega}^n = f_{\omega_{n-1}} \dots f_{\omega_0}$$

Then $(f_{\omega}^n(x))_{n \in \mathbb{Z}}$ is a Markov process. We call the Markov process constructed in this way an Bernoulli IFS associated to $\{f_i\}_{1 \leq i \leq k}$.

We denote the shift map on Ω by

$$\begin{aligned} T_0 : \Omega &\rightarrow \Omega \\ (T_0(\omega))_n &= \omega_{n+1} \end{aligned}$$

and the natural extension on $\mathbb{P} \times M$ as

$$\begin{aligned} T : \Omega \times M &\rightarrow \Omega \times M \\ T(\omega, x) &= (T_0(\omega), f_{\omega_0}(x)) \end{aligned}$$

For any $\omega = (\dots \omega_0 \omega_1 \dots) \in \Omega$, any integer $k, j \in \mathbb{N}$, we denote

$$\begin{aligned} f_{\omega}^{k, k+j} &= f_{\omega_{k+j-1}} \dots f_{\omega_k} \\ f_{\omega}^{k, k-j} &= f_{\omega_{k-j}}^{-1} \dots f_{\omega_{k-1}}^{-1} \end{aligned}$$

Given a Bernoulli IFS $\{f_i\}_{1 \leq i \leq k}$, a probability measure $\mu \in \mathcal{P}(M)$ is called a stationary measure for $\{f_i\}_{1 \leq i \leq k}$ if

$$\mu = \frac{1}{k} \sum_{i=1}^k f_i \mu$$

A stationary measure μ is ergodic if it is not a non-trivial convex combinations of other stationary measures.

For any Bernoulli IFS $\{f_i\}_{1 \leq i \leq k}$, there always exists a stationary measure. When $\{f_i\}_{1 \leq i \leq k}$ have a common invariant measure μ , then μ is a stationary measure for the Bernoulli IFS $\{f_i\}_{1 \leq i \leq k}$.

Similar to the study of deterministic dynamics, we can also define Lyapunov exponents for random dynamics.

Theorem 8. (Theorem 3.2 in [13]) Given $f_1, \dots, f_k \in \text{Diff}^1(M, m)$. There exists a measurable set $\Lambda_0 \subset \Omega \times M$ with $(\mathbb{P} \times m)(\Lambda_0) = 1, T(\Lambda_0) = \Lambda_0$ such that :

(1) For every $(\omega, x) \in \Lambda_0$ there exists a decomposition of $T_x M$

$$T_x M = V^{(1)}(\omega, x) \oplus \dots \oplus V^{(r(x))}(\omega, x)$$

and numbers

$$-\infty < \lambda^{(1)}(x) < \lambda^{(2)}(x) < \dots < \lambda^{(r(x))}(x) < +\infty$$

which depend only on x , such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df_\omega^n(x, \xi)\| = \lambda^{(i)}(x)$$

for all $\xi \in V^{(i)}(\omega, x), 1 \leq i \leq r(x)$. Moreover, $r(x), \lambda^{(i)}(x)$ and $V^{(i)}(\omega, x)$ depend measurably on $(\omega, x) \in \Lambda_0$ and

$$r(f_{\omega_0}(x)) = r(x), \lambda^{(i)}(f_{\omega_0}(x)) = \lambda^{(i)}(x), Df_{\omega_0} V^{(i)}(\omega, x) = V^{(i)}(T(\omega, x))$$

for each $(\omega, x) \in \Lambda_0, 1 \leq i \leq r(x)$.

(2) For each $(\omega, x) \in \Lambda_0$, if $\{\xi_1, \dots, \xi_d\}$ is a basis of $T_x M$ which satisfies that for each $i = 1, \dots, r(x)$ there are exactly $\dim V^{(i)}$ many index j satisfies

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log |Df_\omega^n(x, \xi_j)| = \lambda^{(i)}(x)$$

then for every two non-empty disjoint subsets $P, Q \subset \{1, \dots, d\}$ we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log |\angle(Df_\omega^n E_P, Df_\omega^n E_Q)| = 0$$

where E_P and E_Q denote the subspaces of $T_x M$ spanned by the vectors $\{\xi_i\}_{i \in P}$ and $\{\xi_i\}_{i \in Q}$ respectively.

REMARK 4. In Theorem 3.2 [13], the author considered the non-invertible map defined on $\{1, \dots, k\}^{\mathbb{N}} \times M$ and got a filtration of subspaces. Here we consider the invertible map F and get a splitting. The proof of both theorems are by Oseledec's theorem.

We call the numbers $\lambda^{(1)}(x), \dots, \lambda^{(r(x))}(x)$ the Lyapunov exponents at x .

Now we recall some facts about stable and unstable manifolds of random dynamical systems. We follow the presentations in [8]. More detailed information can be found in [13], Chapter III.

Given $f_1, \dots, f_k \in \text{Diff}^{1+s}(M, m)$ for some $s > 0$, for each $(\omega, x) \in \Omega \times M$, we define

$$\mathcal{W}_\omega^s(x) = \{y | d(f_\omega^n(x), f_\omega^n(y)) \rightarrow 0 \text{ exponentially fast, } n \rightarrow \infty\}$$

For $(\mathbb{P} \times m) - a.e.(\omega, x)$ such that the Lyapunov exponents defined at (ω, x) are not all zero, $\mathcal{W}_\omega^s(x)$ are C^{1+s} -manifolds. Endow $\mathcal{W}_\omega^s(x)$ with induced Riemannian distance, we denote $\mathcal{W}_\omega^s(x, l)$ the l -ball in $\mathcal{W}_\omega^s(x)$. We shall use the absolute continuity of the lamination \mathcal{W}_ω^s (for $\mathbb{P} - a.e.\omega$). More precisely, we have

(AC) For almost all ω the following holds. Assume that restricted to a closed set $K \subset M$, $\{\mathcal{W}_\omega^s(x)\}_{x \in K}$ is a continuous lamination (of constant dimension). Then

let V_1, V_2 be submanifolds of dimension complementary to that of the lamination and transversal to the lamination. Choose $l > 0$ and let

$$K_1 = \{x \in V_1 : \mathcal{W}_\omega^s(x) \text{ is transversal to } V_1, \\ \#(\mathcal{W}_\omega^s(x, l) \cap V_2) = 1 \text{ and this intersection is transversal} \}$$

Then the holonomy map along the stable leaves $p : K_1 \rightarrow V_2$ is absolutely continuous in the sense that it maps measure zero sets to measure zero sets.

To simplify the notations, we introduce the following quantities.

NOTATION 4. Given a Bernoulli IFS $\{f_i\}_{1 \leq i \leq k}$, an integer $b \in [1, d-1]$, for any $x \in M$, any $(d-b)$ -subspace $E \subset T_x M$, any $n \in \mathbb{N}$, we denote

$$C(x, E, n) = \mathbb{E}(\log \sup_{v \in U(E^\perp)} \|P_{Df_\omega^n(x, E)^\perp}(Df_\omega^n(x, v))\|) \\ D(x, E, n) = \mathbb{E}(\log \inf_{u \in U(E)} \|Df_\omega^n(x, u)\|)$$

Hereafter for any subspace of $T_x M$ denoted by G , we define $U(G)$ to be the set of unit vectors in G .

Here $C(x, E, n)$ describes the weakest contraction rate in the directions transverse to E for a typical iteration of length n starting from x ; $D(x, E, n)$ describes the strongest contraction (or the weakest expansion) in E for a typical iteration of length n starting from x .

Definition 20. Given a Bernoulli IFS $\{f_i\}_{1 \leq i \leq k}$, an integer $b \in [1, d-1]$, if there exist $n_0 \geq 1$, $\kappa_1 > 0$, $\kappa_2 \in (-\infty, \kappa_1)$ such that for any $x \in M$, any $(d-b)$ -subspace $E \subset T_x M$, we have

$$(6.1) \quad \frac{1}{n_0} C(x, E, n_0) < -\kappa_1$$

$$(6.2) \quad \frac{1}{n_0} D(x, E, n_0) > -\kappa_2$$

Then we say $\{f_i\}_{1 \leq i \leq k}$ is $(n_0, \kappa_1, \kappa_2, b)$ -uniform.

If there exists $b \in [1, d-1]$ such that $\{f_i\}_{1 \leq i \leq k}$ is $(n_0, \kappa_1, \kappa_2, b)$ -uniform then we say $\{f_i\}_{1 \leq i \leq k}$ is $(n_0, \kappa_1, \kappa_2)$ -uniform.

LEMMA 2. There exists $C_1, \sigma > 0$, such that the following is true. For any $x \in M$, any $(d-b)$ -subspace $E \subset T_x M$, any integer $n \geq 0$, we have

$$\mathbb{E}(\sup_{v \in U(E^\perp)} \|P_{Df_\omega^n(x, E)^\perp}(Df_\omega^n(x, v))\|^\sigma) < C_1 e^{-n\sigma\kappa_1}$$

and

$$\mathbb{E}(\|\inf_{v \in U(E)} Df_\omega^n(x, v)\|^{-\sigma}) < C_1 e^{n\sigma\kappa_2}$$

Proof. The proof is similar to that of Lemma 4 in [8].

By our hypothesis, we have

$$\sup_{\substack{x \in M \\ E \in Gr(T_x M, d-b)}} \mathbb{E}(\log \sup_{v \in U(E^\perp)} \|P_{Df_\omega^{n_0}(x, E)^\perp}(Df_\omega^{n_0}(x, v))\|) < -n_0\kappa_1$$

Then by $e^x = 1 + x + O(x^2)$ when $x \in (-1, 1)$, for small $\sigma > 0$ (depending on n_0 and $\|f_i\|_{C^1}, i = 1, \dots, k$), we have

$$\begin{aligned} & \sup_{\substack{x \in M \\ E \in \text{Gr}(T_x M, d-b)}} \mathbb{E} \left(\sup_{v \in U(E^\perp)} \|P_{Df_\omega^{n_0}(x, E)^\perp}(Df_\omega^{n_0}(x, v))\|^\sigma \right) \\ &= 1 + \sigma \sup_{\substack{x \in M \\ E \in \text{Gr}(T_x M, d-b)}} \mathbb{E}(\log \sup_{v \in U(E^\perp)} \|P_{Df_\omega^{n_0}(x, E)^\perp}(Df_\omega^{n_0}(x, v))\|) + O(\sigma^2) \end{aligned}$$

Thus when $\sigma > 0$ is sufficiently small, we have

$$\sup_{\substack{x \in M \\ E \in \text{Gr}(T_x M, d-b)}} \mathbb{E} \left(\sup_{v \in U(E^\perp)} \|P_{Df_\omega^{n_0}(x, E)^\perp}(Df_\omega^{n_0}(x, v))\|^\sigma \right) < e^{-\sigma n_0 \kappa_1}$$

Then by subadditivity, for each integer $l > 0$ we have

$$\sup_{\substack{x \in M \\ E \in \text{Gr}(T_x M, d-b)}} \mathbb{E} \left(\sup_{v \in U(E^\perp)} \|P_{Df_\omega^{ln_0}(x, E)^\perp}(Df_\omega^{ln_0}(x, v))\|^\sigma \right) < e^{-\sigma l n_0 \kappa_1}$$

Then for some large constant C_1 , we got

$$\sup_{\substack{x \in M \\ E \in \text{Gr}(T_x M, d-b)}} \mathbb{E} \left(\sup_{v \in U(E^\perp)} \|P_{Df_\omega^n(x, E)^\perp}(Df_\omega^n(x, v))\|^\sigma \right) < C_1 e^{-\sigma n \kappa_1}$$

for all $n \in \mathbb{N}$. This proves the first inequality. The second inequality follows from a similar argument. \square

In order to state our main proposition properly, we will have to quantify transversality between linear subspaces and curves on the manifold.

Definition 21. Given any $x \in M$, any hyperplane $E \subset T_x M$, for any $l, \rho, \theta, \beta > 0$, a $C^{1+\beta}$ curve $\mathcal{C} \subset M$ through x is called (l, ρ, θ) -regular with respect to E if the following holds:

- (1) The angle between $T_x \mathcal{C}$ and E is no less than θ .
- (2) The distances (restricted to the curve) between x to both ends of \mathcal{C} are no less than l .
- (3) The $C^{1+\beta}$ -norm of \mathcal{C} is at most ρ .

We say a subset $\mathcal{D} \subset M$ through x is (l, ρ, θ) -regular with respect to E if there exists a C^1 curve $\mathcal{C} \subset \mathcal{D}$ that contains x , and is (l, ρ, θ) -regular with respect to E .

PROPOSITION 7. Let $s > 0$, integer $k \geq 2$. Given k diffeomorphisms $f_i \in \text{Diff}^{1+s}(M)$, $i = 1, \dots, k$. Assume that there exists $n_0, \kappa_1 > 0, \kappa_2 \in (-\infty, \kappa_1)$ such that Bernoulli IFS associated to $\{f_i\}_{1 \leq i \leq k}$ is $(n_0, \kappa_1, \kappa_2)$ -uniform. Then there exists $l, \rho, \theta > 0$ such that for all x in a co-null set in M , any hyperplane $E \subset T_x M$, we have

$$\mathbb{P}(\mathcal{W}_\omega^s(x) \text{ exists and is } (l, \rho, \theta)\text{-regular with respect to } E) > \frac{1}{2}$$

Proof. By definition, there exists an integer $b \in [1, d]$, $\kappa_1 > 0$ and $\kappa_2 \in (-\infty, \kappa_1)$ such that $\{f_i\}_{1 \leq i \leq k}$ is $(n_0, \kappa_1, \kappa_2, b)$ -uniform. Now we denote Λ_0 the subset of $\Omega \times M$ in Theorem 8.

By applying Lemma 2 and Borel-Cantelli lemma, we see that: for all (ω, x) in a co-null set in $\Omega \times M$, the Lyapunov exponents are not all zero. Hence $\mathcal{W}_\omega^s(x)$ are defined for $(\mathbb{P} \times m)$ -a.e. (ω, x) .

For m -a.e. $x \in M$, \mathbb{P} -a.e. ω , we already know that $\mathcal{W}_\omega^s(x)$ exists. The main claim is that we can give uniform lower bound for the regularity of these stable

manifolds relying only on n_0, κ_1, κ_2 . For this purpose, we first establish some *a priori* estimates.

LEMMA 3. *For any $(\omega, x) \in \Lambda_0$, let $i \in [1, c]$ be the smallest integer such that $\dim \oplus_{j=1}^i V^{(j)}(\omega, x) \geq b$ and denote $V^-(\omega, x) = \oplus_{j=1}^i V^{(j)}(\omega, x)$. Then for x in a co-null set of M , for each subspace $E \subset T_x M$ of dimension $(d - b)$, we have*

$$\mathbb{P}(V^-(\omega, x) \cap E \neq \{0\}) = 0$$

Proof. Assume to the contrary that there exists a positive measure set $M_0 \subset M$ such that for all $x \in M_0$, $\mathbb{P} - a.e. \omega$ satisfies that $(\omega, x) \in \Lambda_0$, and there exists a subspace $E \subset T_x M$ of dimension $(d - b)$ such that $\mathbb{P}(V^-(\omega, x) \cap E \neq \{0\}) > 0$. Then for $\mathbb{P} - a.e. \omega$ such that $V^-(\omega, x) \cap E \neq \{0\}$, for any $v \in U(V^-(\omega, x) \cap E)$ we have

$$(6.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df_\omega^n(x, v)\| \leq \lambda^{(i)}(x)$$

Since by the choice of i , we have $\dim \oplus_{j=1}^{i-1} V^{(j)}(\omega, x) < b$. Then $\dim \oplus_{j=i}^{r(x)} V^{(j)}(\omega, x) > d - b = \dim E$. Denote $V' = \oplus_{j=i}^{r(x)} V^{(j)}$, we have $P_{V'}(E) \subsetneq V'$. Moreover, denote $E' = \oplus_{j=1}^{i-1} V^{(j)} \oplus P_{V'}(E)$, we have $E \subset E'$ and as a consequence $\|P_{Df_\omega^n(x, E)^\perp}\| \geq \|P_{Df_\omega^n(x, E')^\perp}\|$. By Gram-Schmidt, there exist a basis of $P_{V'}(E)$ of the form: $\bigcup_{j=i}^{r(x)} \{u_{j,1} + v_{j,1}, u_{j,2} + v_{j,2}, \dots, u_{j,l_j} + v_{j,l_j}\}$. Here for each $j = i, \dots, r(x)$, $\{u_{j,1}, \dots, u_{j,l_j}\}$ is a set of orthonormal vectors in $V^{(j)}$ and $\{v_{j,1}, \dots, v_{j,l_j}\} \subset \oplus_{h=i}^{j-1} V^{(h)}$. Since $\dim P_{V'}(E) < \dim V'$, there exist $j \in [i, r(x)]$ and a vector $u \in V^{(j)}$ orthonormal to $\{u_{j,1}, \dots, u_{j,l_j}\}$. Then by Theorem 8, we have $\lim_{n \rightarrow \infty} \frac{1}{n} \log |\angle(Df_\omega^n(x, u), Df_\omega^n(x, E'))| = 0$. Thus

$$(6.4) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \log \|P_{Df_\omega^n(x, E)^\perp}(Df_\omega^n(x, u))\| \\ & \geq \lim_{n \rightarrow \infty} \frac{1}{n} \log \|P_{Df_\omega^n(x, E')^\perp}(Df_\omega^n(x, u))\| \geq \lambda^{(i)}(x) \end{aligned}$$

Thus by $(n_0, \kappa_1, \kappa_2, b)$ uniform condition, Lemma 2 and Borel-Cantelli lemma, we see that the set of events simultaneously realising (6.3) and (6.4) have zero probability. \square

By Lemma 3, we see that for x in a co-null set in M , for any $E \subset T_x M$ of dimension $(d - b)$, we have $\dim V^-(\omega, x) = b$ and $V^-(\omega, x) \cap E = \{0\}$ for $\mathbb{P} - a.e. \omega$.

Now we show that with large probability, the stable manifolds at x form good angles with a prescribed $(d - b)$ -dimensional subspace.

LEMMA 4. *There exist constants C_0, β such that for all x in a co-null set in M , for any $E \subset T_x M$ of dimension $(d - b)$, for any $\epsilon > 0$, we have*

$$\mathbb{P}(\angle(E, V^-(\omega, x)) \leq \epsilon) \leq C_0 \epsilon^\beta$$

Proof. The proof of this lemma follows exactly that of Corollary 4.(b) in [8]. The push-forward of any graph of linear map from E to E^\perp under Df_ω^n gets exponentially close to that of E with large probability. By combining Lemma 3, this shows that E form small angle with $V^-(\omega, x)$ with small probability. We refer the readers to [8] for details. \square

Similar to $V^-(\omega, x)$ defined in Lemma 3, we define a subspace complementary to $V^-(\omega, x)$ as follows. Let $i \in [1, c]$ be the smallest integer such that $\dim \bigoplus_{j=1}^i V^{(j)}(\omega, x) \geq b$, we denote $V^+(\omega, x) = \bigoplus_{j=i+1}^{r(x)} V^{(j)}(\omega, x)$. By definition, it is clear that $T_x M$ is the direct sum of $V^-(\omega, x)$ and $V^+(\omega, x)$.

Now we define Pesin sets as follows. Take any constants $\bar{\kappa}_1, \bar{\kappa}_2 > 0$ such that $\kappa_2 < \bar{\kappa}_2 < \bar{\kappa}_1 < \kappa_1$ (for instances, $\bar{\kappa}_2 = \frac{2\kappa_2 + \kappa_1}{3}, \bar{\kappa}_1 = \frac{\kappa_2 + 2\kappa_1}{3}$). We denote $\epsilon_0 = \frac{1}{100} \min(\bar{\kappa}_2 - \kappa_2, \kappa_1 - \bar{\kappa}_1)$. For some constant $C, \epsilon > 0$ we denote

$$\Delta_{C, \epsilon}(x) = \{\omega \mid \|Df_\omega^{k, k+j}|V^-\| \leq Ce^{\epsilon k - j\bar{\kappa}_1}, \|Df_\omega^{k, k-j}|V^+\| \leq Ce^{\epsilon k + j\bar{\kappa}_2}, \\ \angle(V^-(T^k(\omega, x)), V^+(T^k(\omega, x))) \geq C^{-1}e^{-k\epsilon}, k \geq 0\}$$

Next we will give a lower bound for the probability of the Pesin set.

LEMMA 5. *For each $\sigma > 0$, there exist $C, \epsilon > 0$ such that for any x in a co-null set of M , we have*

$$\mathbb{P}(\Delta_{C, \epsilon}(x)) > 1 - \sigma$$

Proof. By Lemma 4 and the fact that $V^-(T^k(\omega, x))$ depends only on $\omega_k, \omega_{k+1}, \dots$ and $V^+(T^k(\omega, x))$ depends only on $\omega_{k-1}, \omega_{k-2}, \dots$, we have

$$\mathbb{P}(\angle(V^-(T^k(\omega, x)), V^+(T^k(\omega, x))) \leq C^{-1}e^{-k\epsilon}) \leq C^{-\beta}e^{-k\beta\epsilon}$$

Summing up over $k \geq 1$, we see that the probability of the set of events that violet the last condition in the definition of $\Delta_{C, \epsilon}$ can be make arbitrarily small by making C large.

For each $k \geq 1$, conditioned on $\omega_0, \dots, \omega_{k-1}$, we take an arbitrary subspace $E \subset T_{f_\omega^k(x)} M$ of dimension $(d - b)$. Then for each $j \geq 1$, each event ω (still conditioned on $\omega_0, \dots, \omega_{k-1}$) that violets that first condition for $\Delta_{C, \epsilon}$, i.e. $\|Df_\omega^{k, k+j}|V^-\| > Ce^{\epsilon k - j\bar{\kappa}_1}$, either we have $\angle(V^-(T^{k+j}(\omega, x)), Df_\omega^{k, k+j}(x, E)) < C^{-\frac{1}{2}}e^{-(j+k)\epsilon_0}$, or there exists $v \in U(E^\perp)$ such that $\|P_{Df_\omega^{k, k+j}(x, E)^\perp}(Df_\omega^{k, k+j}(x, v))\| \geq C^{\frac{1}{2}}e^{\epsilon k - (k+j)\epsilon_0 - j\bar{\kappa}_1}$. Choose $\epsilon = 2\epsilon_0$ and $j \geq n_0$, by Lemma 2, the probability of these two events are bounded by $C^{-\frac{\beta}{2}}e^{-(j+k)\beta\epsilon_0}$ and $C_1C^{-\frac{\sigma}{2}}e^{-\sigma(k+j)\epsilon_0}$ respectively. Since $\{e^{-\sigma(j+k)\epsilon_0} + e^{-(j+k)\beta\epsilon_0}\}_{k \geq 1, j \geq 1}$ are summable, we see that the probability of the set of events that violet the first condition for $\Delta_{C, \epsilon}$ can be make arbitrarily small by making C large.

For the second condition, we note that for each $k \in \mathbb{N}, 1 \leq j \leq k$, $V^+(T^{k-j}(\omega, x))$ depends only on $\omega_{k-j-1}, \omega_{k-j-2}, \dots$. For each ω violet the second condition, i.e. $\|Df_\omega^{k, k-j}|V^+\| > Ce^{\epsilon k + j\bar{\kappa}_2}$, we have $\inf_{v \in V^+(T^{k-j}(\omega, x))} \|Df_\omega^{k-j, k}(v)\| < C^{-1}e^{-\epsilon k - j\bar{\kappa}_2}$. Conditioned on $\omega_{k-j-1}, \omega_{k-j-2}, \dots$, by Lemma 2, we see that the probability of the events that violet the second condition for k, j are bounded by $C_2C^{-\sigma}e^{-\sigma(j+k)\epsilon}$. Then by the same reasoning, we can make the probability of violating the second condition arbitrarily small by making C large. This proves the lemma. \square

Now by the construction of stable manifolds in the theory of non-uniformly partially hyperbolic systems, we get the following lemma.

LEMMA 6. For each $C, \epsilon > 0$, there exist constants $l, \rho > 0$ such that for any x in a co-null set of M , any $\omega \in \Delta_{C, \epsilon}(x)$, the injective radius of $\mathcal{W}_\omega^s(x)$ is not less than l and the C^{1+s} -norm of $\mathcal{W}_\omega^s(x, l)$ is at most ρ .

Proof. Choose $\hat{\kappa}_1, \hat{\kappa}_2$ as $\hat{\kappa}_1 = \frac{2\kappa_1 + \kappa_2}{3}, \hat{\kappa}_2 = \frac{\kappa_1 + 2\kappa_2}{3}$. For any $\omega \in \Delta_{C, \epsilon}$, any integer $k \geq 0$, we define the Lyapunov metric $|\cdot|'_k$ on $T_{f_\omega^k(x)}M$ as follows. For any $v \in V^-(T^k(\omega, x))$, we define $|v|'_k = \sum_{j=0}^\infty \|Df_\omega^{k,j}(x, v)\| e^{j\hat{\kappa}_1}$; for any $v \in V^+(T^k(\omega, x))$, we define $|v|'_k = \sum_{j=0}^\infty \|Df_\omega^{k,j}(x, v)\| e^{-j\hat{\kappa}_2}$. Finally, we assume that $V^-(T^k(\omega, x)), V^+(T^k(\omega, x))$ are orthogonal under $|\cdot|'_k$. Then it is direct to check that we can apply Theorem 2.1.1 in [15] to conclude. \square

Now Proposition 7 follows from Lemma 4, 5 and 6. \square

Given an constant $\epsilon > 0$, we defined local stable manifold as $\mathcal{W}_{\omega, \epsilon}^s(x) = \{y | d(f_\omega^n(x), f_\omega^n(y)) < \epsilon, \forall n \geq 1\}$. Similar to the Pesin's theory in the study of iterations of a single diffeomorphism, we can show that $\mathcal{W}_{\omega, \epsilon}^s(x)$ depends on (ω, x) measurably.

The following proposition is the core of this section.

PROPOSITION 8. Let $s > 0$, integer $k \geq 2$. Given an IFS $\{f_i\}_{1 \leq i \leq k}$ consisted of k diffeomorphisms $f_i \in \text{Diff}^{1+s}(M), i = 1, \dots, k$. If there exist $l, \rho, \theta > 0$ such that for any $x \in M$, any hyperplane $E \subset T_x M$, we have

$$\mathbb{P}(\mathcal{W}_\omega^s(x) \text{ exists and is } (l, \rho, \theta)\text{-regular with respect to } E) > \frac{1}{2}$$

Then any invariant closed set for IFS $\{f_i\}_{1 \leq i \leq k}$ having positive volume equals to M . In particular, $\{f_i\}_{1 \leq i \leq k}$ is transitive.

Proof. Assume to the contrary that there exists a closed set $\Gamma \subsetneq M$ that is invariant under f_i for all $1 \leq i \leq k$, and $m(\Gamma) > 0$. Denote Γ' the set of Lebesgue density point of Γ . Since f_i is nonsingular with respect to m for all $1 \leq i \leq k$, we have $f_i(\Gamma') = \Gamma'$ for all $1 \leq i \leq k$. Thus replacing Γ by $\overline{\Gamma'}$, we can assume that for any $x \in \Gamma$, any open set U containing x , we have $m(\Gamma \cap U) > 0$.

Take any $y \in M \setminus \Gamma$, there exists $x \in \Gamma$ such that $d(x, y) = \inf_{z \in \Gamma} d(z, y)$. There exists an open neighbourhood of x , denoted by U , such that : for any $z \in U$, any C^{1+s} -curve \mathcal{C} through z that is (l, ρ, θ) -regular with respect to the hyperplane $E' \subset T_z M$, where E' is obtained from E through a local parallel translation, we have $\mathcal{C} \cap \Gamma^c \neq \emptyset$.

Since f_i preserves m for all $1 \leq i \leq k$, we know that for \mathbb{P} -a.e. ω , local stable manifold $\mathcal{W}_{\omega, \text{loc}}^s(x)$ is defined for m -a.e. x , and depend on x measurably. By Lusin's theorem, we can find a set $\Omega_0 \subset \Omega$ such that

1. $\mathbb{P}(\Omega_0) > \frac{9}{10}$;
2. For each $\omega \in \Omega_0$, there exists $\Gamma_\omega \subset \Gamma$, such that :
 - (2.I) $m(\Gamma \setminus \Gamma_\omega) < \frac{1}{10}m(\Gamma \cap U)$;
 - (2.II) $\mathcal{W}_{\omega, \text{loc}}^s(z)$ is defined everywhere and depends continuously on z restricted to $z \in \Gamma_\omega$;
 - (2.III) For any $z \in \Gamma_\omega$, any open set V containing z , we have $m(\Gamma_\omega \cap V) > 0$.

It is direct to see that there exists $z \in \Gamma \cap U$ such that

$$(6.5) \quad \mathbb{P}(z \in \Gamma_\omega) > \frac{3}{4}$$

Indeed, we denote $\overline{m} = \frac{1}{m(\Gamma \cap U)} m|_{\Gamma \cap U}$ the normalised volume form restricted to $\Gamma \cap U$ (this is well-defined because $m(\Gamma \cap U) > 0$), then

$$(\mathbb{P} \times \overline{m})\{(\omega, z); z \in \Gamma_\omega\} \geq \mathbb{P}(\Omega_0) \times \frac{9}{10} > \frac{3}{4}$$

By Fubini's theorem, there exists $z \in \Gamma \cap U$ that satisfies (6.5).

From the hypothesis of the proposition, we have

$$\mathbb{P}(\mathcal{W}_\omega^s(z) \text{ exists and is } (l, \rho, \theta) - \text{regular with respect to } E') > \frac{1}{2}$$

where $E' \subset T_z M$ is obtained from E through a local parallel translation. Then

$$\mathbb{P}(z \in \Gamma_\omega \cap U, \mathcal{W}_\omega^s(z) \text{ exists and is } (l, \rho, \theta) - \text{regular with respect to } E') > \frac{1}{4}$$

By the choice of U , we see that

$$\mathbb{P}(z \in \Gamma_\omega \cap U, \mathcal{W}_\omega^s(z) \text{ exists and } \mathcal{W}_\omega^s(z) \cap \Gamma^c \neq \emptyset) > \frac{1}{4}$$

By (2.II), for each $\omega \in \Omega_0$ such that $z \in \Gamma_\omega$, there exists an open set V_ω containing z such that: for any $z' \in V_\omega \cap \Gamma_\omega$, $\mathcal{W}_\omega^s(z')$ is defined and $\mathcal{W}_\omega^s(z') \cap \Gamma^c \neq \emptyset$. By (2.III), we have $m(V_\omega \cap \Gamma_\omega) > 0$. By the absolutely continuity of the \mathcal{W}_ω^s lamination, for $\mathbb{P} - a.e. \omega$, there exists a set $K_\omega \subset M$ of positive measure such that for each $w \in K_\omega$, there exists $z' \in V_\omega \cap \Gamma_\omega$ such that $w \in \mathcal{W}_\omega^s(z')$. Then

$$\mathbb{P}(m(w \in \Gamma^c; \text{there exists } z' \in \Gamma \text{ such that } w \in \mathcal{W}_\omega^s(z')) > 0) > 0$$

By Fubini's theorem, we have

$$m(w \in \Gamma^c; \mathbb{P}(\text{there exists } z' \in \Gamma \text{ such that } w \in \mathcal{W}_\omega^s(z')) > 0) > 0$$

While $m - a.e.$ point w is typical for the IFS $\{f_i\}_{1 \leq i \leq k}$, for $m - a.e.$ point w , $\mathbb{P} - a.e. \omega$, the trajectory $\{f_\omega^n(w)\}_{n \in \mathbb{N}}$ accumulates at w . Then there exist $w \in \Gamma^c, z' \in \Gamma$, such that $w \in \mathcal{W}_\omega^s(z')$ and $\liminf_{n \rightarrow \infty} d(w, f_\omega^n(w)) = 0$. Then $\liminf_{n \rightarrow \infty} d(w, f_\omega^n(z')) = 0$. This gives a contradiction since $f_\omega^n(z') \in \Gamma$ for all n and $d(w, \Gamma) > 0$. \square

By combining Proposition 8 with the estimates in [8], we are now ready to deduce Theorem 4.

Proof of Theorem 4. In [8], the authors showed that under the hypothesis of Theorem 4, either $\{f_\alpha\}$ is $(n_0, \kappa_1, \kappa_2)$ -uniform for some $n_0, \kappa_1 > 0, \kappa_2 \in (-\infty, \kappa_1)$, in which case the transitivity follows from Proposition 8 ; or $\{f_\alpha\}_\alpha$ is linearizable, in which case we have ergodicity (see [8]), and hence transitivity. \square

Definition 22. Given any $1 \leq l \leq d, \eta > 0$, a continous l - subspace distribution P on M assigning each point $x \in M$ to $P(x) \in Gr(T_x M, l)$, we say that the IFS $\{f_i\}_{1 \leq i \leq k}$ is η - nontransverse to P if the following holds: for any $x \in M$, any $E \in Gr(T_x M, d - l)$, we have

$$\#\{i \mid Df_i(x, E) \text{ is transverse to } P(f_i(x))\} > (1 - \eta)k$$

The criteria in Proposition 8 can be used to construct transitive IFS, for example the one in the following proposition. We give a detailed proof for the following proposition. Then we will give a sketched proof for a different, yet similar situation.

PROPOSITION 9. Given $s > 0, b \in [1, \frac{d}{2}]$ and $g \in \mathcal{DS}_1^{1+s}(M, m; b)$ having an uniformly dominated splitting

$$TM = E_1 \oplus E_2 \oplus E_3$$

with $\dim(E_1) = \dim(E_3) = b$. Let constants $\bar{\chi}_1, \hat{\chi}_1, \bar{\chi}_2, \hat{\chi}_2 > 0$ satisfy (2.5), (2.6) and (2.7) related to the above splitting. Then there exists $\eta > 0$ depending only on the ratios between $\{\bar{\chi}_i, \hat{\chi}_i\}_{1 \leq i \leq 2}$ and $\|g\|_{C^1}$, such that for any integer $L > 0$ diffeomorphisms $h_1, \dots, h_L \in \text{Diff}^{1+s}(M, m)$ that is η -nontransverse to E_1 and η -nontransverse to E_3 , there exist $K \geq 0, n_0, \kappa_1 > 0, \kappa_2 \in (-\infty, \kappa_1)$ such that the Bernoulli IFS $\{h_i, g^K h_i g^{-K}; 1 \leq i \leq L\}$ is $(n_0, \kappa_1, \kappa_2, b)$ -uniform.

Proof. By (2.5), (2.6), (2.7) in Definition 3, there exists constants $\xi > 0$ and $\bar{\chi}_1, \hat{\chi}_1, \bar{\chi}_2, \hat{\chi}_2 > 0$ that satisfy

$$\xi + \bar{\chi}_i < \hat{\chi}_i \text{ for } i = 1, 2$$

and

$$\sup_{v \in E_i \setminus \{0\}} \frac{\|Dg(v)\|}{\|v\|} < e^{\bar{\chi}_i} < e^{\hat{\chi}_i} < \inf_{u \in E_{i+1} \setminus \{0\}} \frac{\|Dg(u)\|}{\|u\|} \text{ for } i = 1, 2$$

We choose constant $\hat{\chi} > 0$ such that

$$(6.6) \quad e^{-\hat{\chi}} < \inf_{v \in TM \setminus \{0\}} \frac{\|Dg(v)\|}{\|v\|} < \sup_{v \in TM \setminus \{0\}} \frac{\|Dg(v)\|}{\|v\|} < e^{\hat{\chi}}$$

We also denote

$$(6.7) \quad A = \sup_{i \in [1, L]} (\|h_i\|_{C^1}, \|h_i^{-1}\|_{C^1})$$

By hypothesis and the compactness of M , there exists a constant $\tau > 0$ such that for any $x \in M$, any $E \in \text{Gr}(T_x M, d - b)$, we have

$$(6.8) \quad \#\{i \mid \angle(Dh_i(x, E), E_1(h_i(x))) > \tau\} > (1 - \eta)L$$

and

$$(6.9) \quad \#\{i \mid \angle(Dh_i(x, E), E_3(h_i(x))) > \tau\} > (1 - \eta)L$$

It is elementary to see that there exists $C_{1,\tau} > 0$ such that for any $x \in M$, any $E \in \text{Gr}(T_x M, d - b)$ satisfying $\angle(E, E_1(x)) > \tau$, for any $q \in \mathbb{N}^*$ we have

$$(6.10) \quad \sup_{v \in U(E^\perp)} \|P_{(Dg^q(x, E))^\perp}(Dg^q(x, v))\| \leq C_{1,\tau} e^{q\bar{\chi}_1}$$

Similarly, there exists $C_{2,\tau} > 0$, any $x \in M$ such that for any $E \in \text{Gr}(T_x M, d - b)$ satisfying $\angle(E, E_3(x)) > \tau$, for any $q \in \mathbb{N}^*$ we have

$$(6.11) \quad \sup_{v \in U(E^\perp)} \|P_{(Dg^{-q}(x, E))^\perp}(Dg^{-q}(x, v))\| \leq C_{2,\tau} e^{-q\hat{\chi}_2}$$

Take $C_\tau = \max(C_{1,\tau}, C_{2,\tau})$ and let η be a small constant to be determined later. We will see that η can be chosen depending only on the ratios between $\{\bar{\chi}_i, \hat{\chi}_i\}_{1 \leq i \leq 2}$ and $\|g\|_{C^1}$.

Denote $h_{i,j} = g^{Kj} h_i g^{-Kj}$ for $1 \leq i \leq L, 0 \leq j \leq 1$, where K is a large integer whose value will be determined depending solely on η, ξ, C_τ and g .

Given $l \geq 1, \{(i_s, j_s)\}_{s=1, \dots, l}$, we denote

$$h(i_1, j_1; \dots; i_l, j_l) = h_{i_l, j_l} \cdots h_{i_1, j_1}$$

Then

$$h(i_1, j_1; \dots; i_l, j_l) = g^{Kj_l} h_{i_l} g^{K(j_{l-1}-j_l)} \dots g^{K(j_1-j_2)} h_{i_1} g^{-Kj_1}$$

We also denote

$$f(i_1, j_1; \dots; i_l, j_l) = g^{K(j_{l-1}-j_l)} h_{i_{l-1}} \dots g^{K(j_1-j_2)} h_{i_1} g^{-Kj_1}$$

For any infinite sequence $\omega = (\omega_k)_{k \in \mathbb{Z}}$, where $\omega_k = (i_k, j_k)$, any $n \geq 1$, we denote

$$\begin{aligned} h_\omega^n &= h(i_1, j_1; \dots; i_n, j_n) \\ f_\omega^n &= f(i_1, j_1; \dots; i_n, j_n) \end{aligned}$$

It is clear from the expression that $f(i_1, j_1; \dots; i_l, j_l)$ is independent of i_l . Moreover, we have

$$(6.12) \quad f_\omega^{n+1} = g^{K(j_n-j_{n+1})} h_{i_n} f_\omega^n$$

To show that for some $K \geq 0$, the IFS associated to $\{h_i, g^K h_i g^{-K}; 1 \leq i \leq L\}$ is $(n_0, \kappa_1, \kappa_2, b)$ -uniform for some $n_0, \kappa_1 > 0, \kappa_2 \in (-\infty, \kappa_1)$, it is enough to show that: there exist $K \geq 0, \kappa_1 > 0, \kappa_2 \in (-\infty, \kappa_1)$ such that for any $x \in M$, any $E \in \text{Gr}(T_x M, d-b)$, any $n \geq 1$, we have

$$(6.13) \quad \begin{aligned} &\mathbb{E}(\log \sup_{v \in U(E^\perp)} \|P_{Df_\omega^{n+1}(x, E)^\perp}(Df_\omega^{n+1}(x, v))\|) - \mathbb{E}(\log \sup_{v \in U(E^\perp)} \\ &\|P_{Df_\omega^n(x, E)^\perp}(Df_\omega^n(x, v))\|) < -\kappa_1 \end{aligned}$$

and

$$(6.14) \quad \mathbb{E}(\log \inf_{u \in U(E)} \|Df_\omega^{n+1}(x, u)\|) - \mathbb{E}(\log \inf_{u \in U(E)} \|Df_\omega^n(x, u)\|) > -\kappa_2$$

We will detail the proof of the first inequality, the second one follows from a similar argument.

By conditioning on the first $(n-1)$ -iterations, the first inequality follows from the following: there exists $\kappa_1 > 0$, such that for any $x \in M$, any $E \in \text{Gr}(T_x M, d-b)$, any $n \geq 1$, any $\{(i_k, j_k)\}_{1 \leq k \leq n-1}$, the follow is true:

$$(6.15) \quad \begin{aligned} J &:= \mathbb{E}(\log \sup_{v \in U(E^\perp)} \|P_{Df_\omega^{n+1}(x, E)^\perp}(Df_\omega^{n+1}(x, v))\| - \log \sup_{v \in U(E^\perp)} \\ &\|P_{Df_\omega^n(x, E)^\perp}(Df_\omega^n(x, v))\| | \omega_k = (i_k, j_k) \text{ for } 1 \leq k \leq n-1) < -\kappa_1 \end{aligned}$$

For any $(j, p) \in \{0, 1\}^2$, we denote

$$(6.16) \quad \begin{aligned} J_{j,p} &:= \mathbb{E}(\log \sup_{v \in U(E^\perp)} \|P_{Df_\omega^{n+1}(x, E)^\perp}(Df_\omega^{n+1}(x, v))\| - \log \sup_{v \in U(E^\perp)} \\ &\|P_{Df_\omega^n(x, E)^\perp}(Df_\omega^n(x, v))\| | \omega_k = (i_k, j_k), \forall k \in [1, n-1], \omega_n = (*, j), \omega_{n+1} = (*, p)) \end{aligned}$$

Then we have

$$(6.17) \quad J = \frac{1}{4}(J_{0,0} + J_{0,1} + J_{1,0} + J_{1,1})$$

By (6.12), for each $(j, p) \in \{0, 1\}^2$ we have

$$\begin{aligned}
 J_{j,p} &\leq \mathbb{E}(\log \sup_{v \in U(E^\perp)} \frac{\|P_{Df_\omega^{n+1}(x,E)^\perp}(Df_\omega^{n+1}(x,v))\|}{\|P_{Df_\omega^n(x,E)^\perp}(Df_\omega^n(x,v))\|} | \omega_k = (i_k, j_k) \\
 &\quad \forall k \in [1, n-1], \omega_n = (*, j), \omega_{n+1} = (*, p)) \\
 &\leq \mathbb{E}(\log \sup_{v \in U(Df_\omega^n(x,E)^\perp)} \|P_{Df_\omega^{n+1}(x,E)^\perp}(Dg^{K(j-h)} Dh_{i_n}(f_\omega^n(x), v))\| \\
 (6.18) \quad &| \omega_k = (i_k, j_k), \forall k \in [1, n-1], \omega_n = (*, j), \omega_{n+1} = (*, p))
 \end{aligned}$$

For any $x \in M$, any $E \in Gr(T_x M, d-b)$, any $(j, p) \in \{0, 1\}^2$, any $i \in \{1, \dots, L\}$, we denote

$$(6.19) \quad J(j, p, i; x, E) = \log \sup_{v \in U(E^\perp)} \|P_{(Dg^{K(j-p)} Dh_i(x, E))^\perp}(Dg^{K(j-p)} Dh_i(x, v))\|$$

Then by (6.18) and the fact that $f_\omega^n(x)$ and $Df_\omega^n(x, E)$ depend only on $\{(i_k, j_k)\}_{1 \leq k \leq n-1}$ and j_n (but not i_n), we have

$$(6.20) \quad J_{j,p} \leq \sup_{\substack{x \in M \\ E \in Gr(T_x M, d-b)}} \frac{1}{L} \sum_{i=1}^L J(j, p, i; x, E)$$

For any $x \in M$, any $E \in T_x M$, any $j \in \{0, 1\}$, any $i \in [1, L]$, we have

$$(6.21) \quad |J(j, j, i; x, E)| \leq \log A$$

For $(j, p) = (0, 1)$, by the hypothesis that the IFS is η -nontransverse to E_3 , for any $x \in M$, any $E \in Gr(T_x M, d-b)$, there are more than $(1-\eta)L$ many indexes i such that $\angle(Dh_i(x, E), E_3(h_i(x))) > \tau$, for which i we have

$$\sup_{v \in U(E^\perp)} \|(P_{(Dg^{-K} Dh_i(x, E))^\perp}(Dg^{-K} Dh_i(x, v))\| \leq C_\tau e^{-K\hat{\chi}_2} A$$

For all $i \in [1, L]$, we have the trivial bound

$$(6.22) \quad \sup_{v \in U(E^\perp)} \|(P_{(Dg^{-K} Dh_i(x, E))^\perp}(Dg^{-K} Dh_i(x, v))\| \leq e^{K\hat{\chi}} A$$

Thus for any $x \in M$, any $E \in Gr(T_x M, d-b)$, we have

$$(6.23) \quad \frac{1}{L} \sum_{i=1}^L J(0, 1, i; x, E) \leq \eta K \hat{\chi} + (1-\eta)(-K\hat{\chi}_2 + \log C_\tau) + \log A$$

Similarly, for any $x \in M$, any $E \in Gr(T_x M, d-b)$, we have

$$(6.24) \quad \frac{1}{L} \sum_{i=1}^L J(1, 0, i; x, E) \leq \eta K \hat{\chi} + (1-\eta)(K\hat{\chi}_1 + \log C_\tau) + \log A$$

By (6.17), (6.20), (6.21), (6.23) and (6.24), we have

$$\begin{aligned}
 J &\leq \log A + \frac{1}{2} \eta K \hat{\chi} + \frac{1}{4} ((1-\eta)K(\hat{\chi}_1 - \hat{\chi}_2) + 2 \log C_\tau) \\
 &\leq \log A + \frac{1}{2} \log C_\tau + (\frac{1}{2} \eta \hat{\chi} - \frac{1}{2} (1-\eta) \xi) K
 \end{aligned}$$

Thus for any $\tilde{\kappa}_1 < \frac{1}{2}\zeta$, by letting η to be sufficiently small depending only on $\hat{\chi}, \zeta, \tilde{\kappa}_1$, and by letting K to be sufficiently large depending on η, ζ, C_τ, A , we get (6.15) for

$$\kappa_1 = K\tilde{\kappa}_1 > 0$$

This completes the proof. \square

PROPOSITION 10. *Given $s > 0$, $g \in \mathcal{DS}_2^{1+s}(X, m)$. Let the dominated splitting $TM = E_1 \oplus E_2$ and constants $\tilde{\chi}_1, \hat{\chi}_1, \chi^u, \chi^s$ be given by Definition 10. Then there exists $\eta > 0$ depending only on the ratios between $\tilde{\chi}_1, \hat{\chi}_1, \chi^u, \chi^s$ such that for any $L > 0$ diffeomorphisms $h_1, \dots, h_L \in \text{Diff}^{1+s}(M, m)$ that is η -nontransverse to E_1 and η -nontransverse to E_2 , there exist integers $K, n_0 > 0$ and $\kappa_1 > 0, \kappa_2 \in (-\infty, \kappa_1)$ such that the Bernoulli IFS $\{h_i, g^K h_i g^{-K}; 1 \leq i \leq L\}$ is $(n_0, \kappa_1, \kappa_2, b)$ -uniform.*

Proof. The main idea of the proof is the same as Proposition 9. We will only detail the main differences.

We define A as in (6.7) and take constant $\chi^u, \chi^s > 0$ such that

$$e^{-\chi^s} < \inf_{v \in TM \setminus \{0\}} \frac{\|Dg(v)\|}{\|v\|} < \sup_{v \in TM \setminus \{0\}} \frac{\|Dg(v)\|}{\|v\|} < e^{\chi^u}$$

and we denote

$$\hat{\chi} = \max(\chi^s, \chi^u)$$

Similar to (6.8), (6.9), we take $\tau > 0$ such that (6.8) holds and (6.9) holds with E_2 in place of E_3 .

Similar to (6.10), for some large $D_{1,\tau} > 0$, for any $x \in M$, any $E \in \text{Gr}(T_x M, d - \dim E_1)$ satisfying $\angle(E, E_1(x)) > \tau$ and any $q \in \mathbb{N}^*$ we have

$$\begin{aligned} \sup_{v \in U(E^\perp)} \|P_{(Dg^q(x, E))^\perp}(Dg^q(x, v))\| &\leq D_{1,\tau} e^{q\tilde{\chi}_1} \\ \inf_{v \in U(E)} \|Dg^q(x, v)\| &\geq D_{1,\tau}^{-1} e^{q\hat{\chi}_1} \end{aligned}$$

Since by hypothesis, we have $\dim E_2 = d - \dim E_1 \geq \dim E_1$. Thus for any $E \in \text{Gr}(T_x M, d - \dim E_1)$ transversal to E_2 , we have $E + E_2 = T_x M$. Then for some large $D_{2,\tau}$, for any $x \in M$, any $E \in \text{Gr}(T_x M, d - \dim E_1)$ satisfying $\angle(E, E_2(x)) > \tau$ and any $q \in \mathbb{N}^*$, we have

$$\sup_{v \in U(E^\perp)} \|P_{(Dg^{-q}(x, E))^\perp}(Dg^{-q}(x, v))\| \leq D_{2,\tau} e^{-q\hat{\chi}_1}$$

Take $D_\tau = \max(D_{1,\tau}, D_{2,\tau})$.

Again we are lead to verify (6.13) and (6.14). For (6.13), all the rest of the arguments in proof of Proposition 9 carry through with $\hat{\chi}_1$ in place of $\hat{\chi}_2$. More precisely, we obtain that

$$\begin{aligned} (6.25) \quad &\mathbb{E}(\log \sup_{v \in U(E^\perp)} \|P_{Df_\omega^{n+1}(x, E)^\perp}(Df_\omega^{n+1}(x, v))\|) - \mathbb{E}(\log \sup_{v \in U(E^\perp)} \\ &\|P_{Df_\omega^n(x, E)^\perp}(Df_\omega^n(x, v))\|) < \log A + \frac{1}{2}\eta K \hat{\chi} + \frac{1}{4}((1 - \eta)K(\tilde{\chi}_1 - \hat{\chi}_1) + 2 \log D_\tau) \end{aligned}$$

For any $\tilde{\kappa}_1 < \frac{1}{4}(\hat{\chi}_1 - \tilde{\chi}_1)$, by letting η to be sufficiently small depending only on the ratios between $\hat{\chi}, \hat{\chi}_1, \tilde{\chi}_1 \tilde{\kappa}_1$, and by letting K to be sufficiently large depending on η, ζ, C_τ, A , we get (6.13) for $\kappa_1 = K\tilde{\kappa}_1$.

We will show (6.14). We need to use hypothesis (3.1) to ensure that κ_2 can be chosen to be smaller than κ_1 .

For any $x \in M$, $E \subset T_x M$, any $n \geq 1$, any $\{(i_k, j_k)\}_{1 \leq k \leq n-1}$, we define

$$H = \mathbb{E}(\log \inf_{u \in U(E)} \|Df_\omega^{n+1}(x, u)\|) - \log \inf_{u \in U(E)} \|Df_\omega^n(x, u)\| \|\omega_k = (i_k, j_k) \text{ for } 1 \leq k \leq n-1\|$$

For any $(j, p) \in \{0, 1\}^2$ and integer $i \in [1, L]$, we denote

$$(6.26) \quad H_{j,p} := \mathbb{E}(\log \inf_{u \in U(E)} \|Df_\omega^{n+1}(x, v)\| - \log \inf_{u \in U(E)} \|Df_\omega^n(x, u)\| \|\omega_k = (i_k, j_k), \forall k \in [1, n-1], \omega_n = (*, j), \omega_{n+1} = (*, p)\|)$$

$$(6.27) \quad H(j, p, i; x, E) = \log \inf_{u \in U(E)} \|Dg^{K(j-p)} Dh_i(x, u)\|$$

Similar to (6.17) and (6.20), we have

$$(6.28) \quad H = \frac{1}{4}(H_{0,0} + H_{0,1} + H_{1,0} + H_{1,1})$$

and

$$(6.29) \quad H_{j,p} \geq \inf_{\substack{x \in M \\ E \in Gr(T_x M, d - \dim E_1)}} \frac{1}{L} \sum_{i=1}^L H(j, p, i; x, E)$$

Similar to (6.21) and (6.22), we have trivial bounds

$$(6.30) \quad H(j, j, i; x, E) \geq -\log A$$

$$(6.31) \quad H(j, p, i; x, E) \geq -K\hat{\chi} - \log A$$

for all $(j, p) \in \{0, 1\}^2$ and $1 \leq i \leq L$.

For $(j, p) = (0, 1)$, for any $x \in M$, $E \in Gr(T_x M, d - \dim E_1)$, any integer $i \in [1, L]$, we have

$$(6.32) \quad H(0, 1, i; x, E) \geq -K\chi'' - \log A$$

For $(j, p) = (1, 0)$, by the hypothesis that the IFS is η -nontransverse to E_1 , for any $x \in M$, any $E \in Gr(T_x M, d - \dim E_1)$, there are more than $(1 - \eta)L$ many indexes i such that $\angle(Dh_i(x, E), E_1(h_i(x))) > \tau$, for which i we have

$$(6.33) \quad \inf_{u \in U(E)} \|Dg^K Dh_i(x, u)\| \geq D_\tau^{-1} e^{K\hat{\chi}_1} A^{-1}$$

Put together (6.28) to (6.33), we obtain that

$$(6.34) \quad \begin{aligned} \text{LHS of (6.14)} &\geq -\log A - \frac{1}{4} \log D_\tau - \frac{1}{4} K(\chi'' + \eta\hat{\chi} - (1 - \eta)\hat{\chi}_1) \\ &\geq \frac{K}{4}(\hat{\chi}_1 - \chi'') + O(\eta K\hat{\chi}) + O(\log A + \log D_\tau) \end{aligned}$$

Thus for any $\tilde{\kappa}_2 > \frac{1}{4}(\chi'' - \hat{\chi}_1)$, by letting η to be sufficiently small depending on the ratios between $\hat{\chi}_1, \chi'', \hat{\chi}, \tilde{\kappa}_2$, and by letting K to be sufficiently large depending on $A, D_\tau, \eta, \tilde{\kappa}_2$, we have

$$\text{LHS of (6.14)} > K\tilde{\kappa}_2 =: \kappa_2$$

Finally, by (3.1) we see that $\hat{\chi}_1 - \bar{\chi}_1 > \chi'' - \hat{\chi}_1$, hence we can choose $\tilde{\kappa}_2 < \tilde{\kappa}_1$, and hence $\kappa_2 < \kappa_1$. This completes the proof. \square

6.2. Creating transversality. Recall that $X = Y \times N$ where Y, N are defined in Notation 1, and π_Y (resp. π_N) is the canonical projection $X \rightarrow Y$ (resp. $X \rightarrow N$).

Given an integer $l \in [1, c]$, we define a smooth Riemannian manifold, denoted by Gr_l , as follows. For each $x \in X$, we choose a local chart containing $\pi_Y(x)$ and a local chart containing $\pi_N(x)$, the product of these two form a local (central) foliation chart containing x . This allows us to assign a smooth manifold fibered over X whose fibre is identified with the l -Grassmannian of the centre subspace. We denote this fiber bundle by Gr_l and denote $p : Gr_l \rightarrow X$. An element of Gr_l will be denoted by (x, E) for some $x \in X$ and $E \in Gr(E_F^c(x), l)$. We fix an arbitrary Riemannian metric on Gr_l .

In the rest of this subsection, we fix a skew product map F belonging to either $\mathcal{U}_1^r(X, Vol; l)$ for some integer $l \in [1, \frac{c}{2}]$ or $\mathcal{U}_2^r(X, Vol)$. We consider the lift of $F : X \rightarrow X$ to the Gr_l , denoted by $G(F) : Gr_l \rightarrow Gr_l$, where for any $(x, E) \in Gr_l$

$$G(F)(x, E) = (F(x), DF(x, E))$$

Using the invariant splitting of F , we define a $G(F)$ -invariant splitting of the tangent space of Gr_l as follows.

Definition 23. For any $(x, E) \in Gr_l$ (E denotes a l -subspace of $E_F^c(x)$), the tangent space of Gr_l at (x, E) can be canonically identified with

$$T_{(x, E)}Gr_l = \{(v, \phi) | v \in T_x C, \phi : E \rightarrow E^\perp\}$$

Here E^\perp is defined to be the orthogonal complement of E in $E_F^c(x)$. For $* = s, u$, we denote

$$E_{G(F)}^*(x, E) = \{(v, 0); v \in E_x^*\}$$

and denote

$$E_{G(F)}^c(x, E) = \{(v, \phi); v \in E_F^c(x), \phi : E \rightarrow E^\perp\}$$

We have

$$(6.35) \quad T_{(x, E)}Gr_l = E_{G(F)}^s(x, E) \oplus E_{G(F)}^c(x, E) \oplus E_{G(F)}^u(x, E)$$

For each $(x, E) \in Gr_l$, we define a norm $\|\cdot\|_*$ on $T_{(x, E)}Gr_l$ as follows. For each $(v, \phi) \in T_{(x, E)}Gr_l$, we denote

$$\|(v, \phi)\|_* = \|v\| + \|\phi\|$$

It is direct to see that $\|\cdot\|_*$ is a Finsler metric on Gr_l and is equivalent to the Riemannian metric on Gr_l .

LEMMA 7. Given integer $r \geq 3$, for any C^r skew product map F , we have

- (1) If F is 1-center bunching, then $G(F)$ is a C^{r-1} partially hyperbolic system foliated by compact central leaves. Each compact central leaf of $G(F)$ is of the form $p^{-1}(N_y)$ for some $y \in Y$.
- (2) If $F \in \mathcal{U}_1^r(X, Vol; l)$ for some $l \in [1, \frac{c}{2}]$ or $F \in \mathcal{U}_2^r(X, Vol)$, for any compact central leaves C_1, C_2 belonging to \mathcal{W}_F^{cu} (resp. \mathcal{W}_F^{cs}), $H_{G(F), p^{-1}(C_1), p^{-1}(C_2)}^u$ (resp. $H_{G(F), p^{-1}(C_1), p^{-1}(C_2)}^s$) are well-defined and C^1 .

- (3) If F is 1-center bunching, given \hat{F} any C^r deformation of F , after possibly reducing the size of U , the map $\widehat{\mathbb{G}}(F) : U \times Gr_l \rightarrow Gr_l$ defined as

$$\widehat{\mathbb{G}}(F)(b, x, E) = \mathbb{G}(\hat{F}(b, \cdot))(x, E)$$

is a C^{r-1} deformation of $\mathbb{G}(F)$.

Proof. We first prove (1), the rest follows from the standard theory of normal hyperbolic systems.

Since F is C^r , $\mathbb{G}(F)$ is a C^{r-1} diffeomorphism. It is direct to check that $\mathbb{G}(F)$ expand $E_{\mathbb{G}(F)}^u$ (resp. contract $E_{\mathbb{G}(F)}^s$) at rates stronger than $e^{\tilde{\chi}^u}$ (resp. stronger than $e^{-\tilde{\chi}^s}$). Moreover, $\mathbb{G}(F)$ expands $E_F^c \times \{0\}$ at a rate weaker than $e^{\hat{\chi}^c}$ and contracts it at a rate weaker than $e^{\tilde{\chi}^c}$ with respect the metric $\|\cdot\|_*$.

For any $(x, E) \in Gr_l$, any $(0, \phi) \in T_{(x,E)}Gr_l$, we denote $(0, \psi) = D\mathbb{G}(F)_{(x,E)}(0, \phi)$. By definition and straightforward computations, we got

$$\psi(u) = P_{DF(x,E)^\perp}(DF(x, \phi(DF^{-1}(F(x), u))))$$

Moreover, we have

$$\begin{aligned} \|\psi\| &= \sup_{u \in DF(x,E)} \frac{\|\psi(u)\|}{\|u\|} \\ &= \sup_{u_0 \in E} \frac{\|\psi(DF(x, u_0))\|}{\|DF(x, u_0)\|} \\ &= \sup_{u_0 \in E} \inf_{v_0 \in E} \frac{\|DF(x, \phi(u_0)) - DF(x, v_0)\|}{\|DF(x, u_0)\|} \\ &\leq e^{-\tilde{\chi}^c + \hat{\chi}^c} \sup_{u_0 \in E} \frac{\|\phi(u_0)\|}{\|u_0\|} \\ &= e^{-\tilde{\chi}^c + \hat{\chi}^c} \|\phi\| \end{aligned}$$

Similarly, we have

$$\|\psi\| \geq e^{\tilde{\chi}^c - \hat{\chi}^c} \|\phi\|$$

If F is assumed to be 1-center bunching, we have

$$\begin{aligned} -\tilde{\chi}^s &< \tilde{\chi}^c - \hat{\chi}^c \\ \tilde{\chi}^u &> -\tilde{\chi}^c + \hat{\chi}^c \end{aligned}$$

Hence $\mathbb{G}(F)$ is partially hyperbolic with respect to the Finsler metric $\|\cdot\|_*$, hence is also partially hyperbolic with respect to the Riemannian metric on Gr_l . It is direct to check that the centre leaves of $\mathbb{G}(F)$ are of the forms $p^{-1}(\mathcal{C})$ where \mathcal{C} is a compact centre leaf of X , and they form a C^{r-1} foliation of Gr_l (i.e. there exists C^{r-1} foliation charts). This completes the proof of (1).

Now we have shown that $\mathbb{G}(F)$ is dynamically coherent with compact central leaves. Let \mathcal{C} be an arbitrary compact central leaf of F , we denote $\hat{\mathcal{C}} = p^{-1}(\mathcal{C})$, which is a compact central leaf of $\mathbb{G}(F)$. For $\mathcal{C}_1, \mathcal{C}_2$, two arbitrary compact central leaf of F contained in the same \mathcal{W}^{cu} (resp. \mathcal{W}^{cs}), We have a homeomorphism $H_{\mathbb{G}(F), \hat{\mathcal{C}}_1, \hat{\mathcal{C}}_2}^u$ (resp. $H_{\mathbb{G}(F), \hat{\mathcal{C}}_1, \hat{\mathcal{C}}_2}^s$) between $\hat{\mathcal{C}}_1, \hat{\mathcal{C}}_2$.

If F belongs to either $\mathcal{U}_1^r(X, Vol; l)$ for some integer $l \in [1, \frac{c}{2}]$ or $\mathcal{U}_2^r(X, Vol)$, we always have

$$(6.36) \quad -\bar{\chi}^{s,u} + \hat{\chi}^c - \bar{\chi}^c + \max(\hat{\chi}^c, 0) + \max(-\bar{\chi}^c, 0) < 0$$

Then F is 1-center bunching. Moreover, $\mathbb{G}(F)$ is also 1-center bunching. Indeed, in the case where $\bar{\chi}^c \leq 0 \leq \hat{\chi}^c$, the rate of expansion/contraction of $D(\mathbb{G}(F))$ in the centre directions is between $[-\bar{\chi}^c + \bar{\chi}^c, \hat{\chi}^c - \bar{\chi}^c]$. Then in this case, (6.36) gives

$$-\bar{\chi}^{s,u} + (\hat{\chi}^c - \bar{\chi}^c) - (-\bar{\chi}^c + \bar{\chi}^c) < 0$$

which is exact the 1-center bunching condition for $\mathbb{G}(F)$; in the case where $\bar{\chi}^c \leq \hat{\chi}^c < 0$, the rate of expansion/contraction of $D(\mathbb{G}(F))$ in the centre directions is between $[\bar{\chi}^c, \hat{\chi}^c - \bar{\chi}^c]$. Then in this case, (6.36) gives

$$-\bar{\chi}^{s,u} + (\hat{\chi}^c - \bar{\chi}^c) - \bar{\chi}^c < 0$$

which is again the 1-center bunching condition for $\mathbb{G}(F)$ in this case; Similarly, when $0 < \bar{\chi}^c \leq \hat{\chi}^c$, we can obtain the 1-center bunching condition for $\mathbb{G}(F)$ from (6.36). Thus (2) follows from (1) and Theorem 6.

Finally, it is obvious that $\widehat{\mathbb{G}(F)}(0, \cdot) = \mathbb{G}(F)$. By (1) and the fact that partially hyperbolic systems form an C^1 -open set of diffeomorphisms, we obtained (3). \square

A construction similar to (3) in Lemma 7 can be carried out for infinitesimal C^r deformations. In order to construct the desired perturbation, we need more detailed knowledge of a lift of a smooth vector field on a central leaf \mathcal{C} . The following lemma link the vector field on \mathcal{C} with the vector field on $p^{-1}(\mathcal{C})$.

LEMMA 8. *Let \mathcal{C} be a compact central leaf, and $\mathcal{D} = p^{-1}(\mathcal{C})$. (\mathcal{D} is naturally isomorphic to $Gr(\mathcal{C}, l)$). Let V be a C^r vector field on \mathcal{C} . Let $\Phi_V : \mathcal{C} \times \mathbb{R} \rightarrow \mathcal{C}$ be the flow generated by V , i.e. $\partial_t \Phi_V(x, t) = V(\Phi_V(x, t))$ for all $(x, t) \in \mathcal{C} \times \mathbb{R}$. Define map $\mathbb{G}(\Phi_V)$ as*

$$\begin{aligned} \mathbb{G}(\Phi_V) : \mathcal{D} \times \mathbb{R} &\rightarrow \mathcal{D} \\ (x, E, t) &\mapsto \mathbb{G}(\Phi_V(\cdot, t))(x, E) \end{aligned}$$

where $\mathbb{G}(\Phi_V(\cdot, t))$ is the lift of the map $\Phi_V(\cdot, t)$ to \mathcal{D} . Then $\mathbb{G}(\Phi_V)$ is the flow generated by $\mathbb{G}(V)$, a C^{r-1} vector field on \mathcal{D} . Here $\mathbb{G}(V)$ is defined as follows. For each $(x, E) \in \mathcal{D}$,

$$(6.37) \quad \mathbb{G}(V)(x, E) = (V(x), \phi)$$

where $\phi \in \mathcal{L}(E, E^\perp)$

$$(6.38) \quad \phi(u) = P_{E^\perp}(DV(x, u)), \forall u \in E$$

Proof. Check by definitions. \square

In order to construct a suitable smooth deformation, we need to find moderately separated loops contained in a small region. This is summarised in the following lemma.

LEMMA 9. *For any $L > 0$, there exists $C(L) > 0$, such that for any central leaf \mathcal{C} , any $\sigma > 0$, there exist L su-loops at \mathcal{C} , denoted by $\gamma_1, \dots, \gamma_L$, where $\gamma_i = (\mathcal{C}_{i,1}, \mathcal{C}_{i,2}, \mathcal{C}_{i,3})$, such that the parts of the stable/unstable leaves connecting the central leaves are contained in $B(\mathcal{C}, \sigma)$ and*

$$(6.39) \quad \min_{\substack{1 \leq i, j \leq L, i \neq j \\ 1 \leq k \leq 3}} (d(\mathcal{C}, \mathcal{C}_{i,1}), d(\mathcal{C}_{i,1}, \mathcal{C}_{j,k}), d(\mathcal{C}_{i,1}, \mathcal{C}_{i,2}), d(\mathcal{C}_{i,1}, \mathcal{C}_{i,3})) \geq 3C(L)^{-1}\sigma$$

Proof. Without loss of generality, we assume that $\sigma > 0$ is small enough so that for any $y \in Y$, the map \exp_y restricted to $B_y(0, \sigma) \subset T_y Y$ is a 2-Lipschitz diffeomorphism. We note that $\exp_y(E^*(y) \cap B_y(0, \sigma))$ is $o(\sigma)$ -close to $\mathcal{W}_F^*(y)$ for $* = s, u$. The lemma follows from the above observations and the fact that the angles between stable unstable distributions are uniformly lower bounded. \square

PROPOSITION 11. *Given any $F \in \mathcal{U}_1^r(X, \text{Vol}; l)$ for some integer $l \in [1, \frac{c}{2}]$ (resp. $\mathcal{U}_2^r(X, \text{Vol})$), for any $\eta \in (0, 1)$, there exists an integer $L > 0$, an integer $q_0 > 0$ such that the following is true. For any periodic central leaf \mathcal{C} with period $q > q_0$, there exist L su-loops at \mathcal{C} , denoted by $\gamma_1, \dots, \gamma_L$, such that for any $\epsilon > 0$, there exists $F' \in \mathcal{U}_1^r(X, \text{Vol}; l)$ (resp. $F' \in \mathcal{U}_2^r(X, \text{Vol})$) satisfying the following*

(1) $d_{C^r}(F', F) < \epsilon$;

(2) The IFS $\{H_{F', \gamma_i}\}_{1 \leq i \leq L} \subset \text{Diff}^{1+}(\mathcal{C}, m)$ is η -nontransverse to $E_1|_{\mathcal{C}}, E_3|_{\mathcal{C}}$ (resp. η -nontransverse to $E_1|_{\mathcal{C}}, E_2|_{\mathcal{C}}$). Moreover, $F'^q : \mathcal{C} \rightarrow \mathcal{C}$ preserves the splitting $E_1|_{\mathcal{C}} \oplus E_2|_{\mathcal{C}}$.

Proof. First we consider the case where $F \in \mathcal{U}_1^r(X, \text{Vol}; l)$ for some integer $l \in [1, \frac{c}{2}]$.

Construct Gr_{c-l} as in the beginning of this section and denote the canonical projection by $p : Gr_{c-l} \rightarrow X$. Denote $\mathcal{D} = p^{-1}(\mathcal{C})$. By construction \mathcal{D} is the $(c-l)$ -Grassmannian bundle over \mathcal{C} . It is direct to calculate that

$$\dim(\mathcal{D}) = c + (c-l)l$$

In the following, we denote $c_l = \dim(\mathcal{D})$.

By Definition 11 and 9, we see that the map $F^q|_{\mathcal{C}} \in \mathcal{DS}_1^1(N, m; l)$ modulo the natural isomorphism between \mathcal{C} and N and we have a natural isomorphism

$$Gr_{c-l}(X) = Y \times Gr(N, c-l)$$

Moreover, the restriction of $E_1 \oplus E_2 \oplus E_3$ to \mathcal{C} gives the splitting in the definition of $\mathcal{DS}_1^1(N, m; l)$.

We define

$$\Sigma = \{(x_i, F_i)_{i=1}^L \in \mathcal{D}^L \mid \#\{i; F_i \text{ is not transverse to } E_1(x_i) \text{ or } E_3(x_i)\} \geq \eta L\}$$

We now estimate the Hausdorff dimension of Σ . For $i = 1, 3$, we consider the map $p_i : \mathcal{C} \rightarrow Gr(\mathcal{C}, l)$ defined by

$$p_i(x) = (x, E_i(x))$$

By (3.2) in Definition 11, $F^q|_{\mathcal{C}}$ along with the splitting $\oplus_{i=1}^3 E_i|_{\mathcal{C}}$ is $\frac{c}{c+1}$ -pinching. By Proposition 1 and its corollary, p_1, p_3 is θ -Holder for some $\theta \in (\frac{c}{c+1}, 1)$. We denote

$$\Sigma_0 = \{(x, E) \in \mathcal{D} \mid E \text{ is not transverse to } E_1(x) \text{ or } E_3(x)\}$$

LEMMA 10. *We have*

$$(6.40) \quad HD(\Sigma_0) < \dim(\mathcal{D})$$

Proof. Take any $\beta \in (\theta^{-1}, \frac{c+1}{c})$. For small $\delta > 0$, we choose a δ^β -net in \mathcal{C} , denoted by \mathcal{N} , such that $\#\mathcal{N} = O(\delta^{-c\beta})$. For each $x \in \mathcal{N}$, the subset of $Gr(T_x \mathcal{C}, c-l)$ defined as $A(x) = \{E \in Gr(T_x \mathcal{C}, c-l) \mid E \text{ is not transverse to } E_1(x) \text{ and } E_3(x)\}$ has dimension at most $l(c-l) - 1$. Thus A can be covered by $O(\delta^{-l(c-l)+1})$ many δ -balls. For each $y \in \mathcal{C}$, there exists $x \in \mathcal{N}$ such that $d(x, y) < \delta^\beta$, thus $d((x, E_1(x)), (y, E_1(y))) \lesssim \delta^{\theta\beta}$. Since $\theta\beta > 1$, when δ is sufficiently small, we

can choose a \mathcal{N}_1 , a subset of $\bigcup_{x \in \mathcal{N}} A(x)$, such that : 1. $\#\mathcal{N}_1 = O(\delta^{-c\beta-l(c-l)+1})$;
2. \mathcal{N}_1 form a δ -net in Σ_0 . By the choice of β , we see that $c\beta - 1 + l(c-l) < c + l(c-l) = \dim(\mathcal{D})$. Since δ can be arbitrarily small, this implies that $HD(\Sigma_0) \leq c\beta - 1 + l(c-l) < \dim(\mathcal{D})$. \square

We take $g > 0$ such that $HD(\Sigma_0) < \dim(\mathcal{D}) - g$.

By the definition of Σ , we have

$$\Sigma = \bigcup_{\substack{L \geq k \geq \eta L \\ \{i_1, \dots, i_k\} \subset [L]}} \prod_{j \in \{i_1, \dots, i_k\}} \Sigma_0 \times \prod_{j \notin \{i_1, \dots, i_k\}} \mathcal{D}$$

Then we have

$$HD(\Sigma) < \dim(\mathcal{D}^L) - \eta Lg$$

We choose $L = L(\eta, g, c)$ so that

$$\eta Lg > c_l = \dim(\mathcal{D})$$

We now construct a collection of vector fields on N which will be used as building blocks for the construction of the infinitesimal C^r deformation V .

We fix a metric on \mathcal{D} . For any $p \in N, E \in Gr(T_p N, l)$, any $(v, \phi) \in T_{(x,E)} \mathcal{D}$ where $v \in T_x \mathcal{C}, \phi \in \mathcal{L}(E, E^\perp)$, there exists a divergence free vector field on N , denoted by V_0 , such that

$$V_0(x) = v \text{ and } \phi(u) = P_{E^\perp}(DV_0(x, u)) \text{ for all } u \in E$$

By Lemma 8, $\mathbb{G}(\Phi^{V_0})$ is generated by the flow $\mathbb{G}(V_0)$, and by our choice, $\mathbb{G}(V_0)(x, E) = (v, \phi)$.

We can find a finite collection of points in $T\mathcal{D}$, denoted by $\{(x_\alpha, E_\alpha, v_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ such that the set of vector fields V_α we constructed accordingly satisfy the following property. There exists $\kappa > 0$ such that for each $(x, E) \in \mathcal{D}$, there exists a set of indexes $\alpha_1, \dots, \alpha_{c_l}$ such that

$$\det((\mathbb{G}(V_{\alpha_i})(x, E))_{1 \leq i \leq c_l}) > \kappa$$

There exists a constant $C_1 > 0$, such that the following holds. For any $\sigma > 0$, any $y \in Y$, there exists a smooth function ρ on Y supported in $B(y, \sigma)$ such that

$$\rho(y) = 1 \text{ and } Lip(\rho) \left\| \sum_{\alpha \in \mathcal{A}} V_\alpha \right\|_{C^0} < C_1 \sigma^{-1}$$

Let $C_2 = C(L)$ be given by Lemma 9. Let $q_0 = R_0(L, C_1 C_2, \frac{1}{2}\kappa)$ be given by Lemma 1. We take any f -periodic point $y \in Y$ with period larger than q_0 . Then for some sufficiently small $\sigma > 0$, denote $Q_0 = \pi_Y^{-1}(B(y, \sigma))$ and $Q = p^{-1}(Q_0)$, we can ensure that $R(Q) > q_0$. Then by Lemma 9, we can find L su -loops at \mathcal{C} denoted by $\gamma_1, \dots, \gamma_L$, whose lifts $\Gamma_1, \dots, \Gamma_L$ are connected by stable/unstable manifolds contained in Q and satisfy (6.39) for C_2 . Here the lift of a su -loop for F is defined as follows. For any su -loop for $F, \gamma = (C_1, C_2, C_3)$, we can get a su -loop for $\mathbb{G}(F)$ at \mathcal{D} , denoted by $\Gamma = (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$, where $\mathcal{D}_i = p^{-1}(C_i)$ for $i = 1, 2, 3$.

Denote $I = |\mathcal{A}|L$. We construct V , an infinitesimal C^r deformation with I -parameters, as follows. By the choice of C_1 , for each $i = 1, \dots, L$, we can choose a bump function ρ_i on Y , supported in $\pi_Y(B(C_{i,1}, C_2^{-1}\sigma))$ such that

$$(6.41) \quad \rho_i \pi_Y|_{C_{i,1}} = 1 \text{ and } Lip(\rho_i) \left\| \sum_{\alpha \in \mathcal{A}} V_\alpha \right\|_{C^0} < C_1 C_2 \sigma^{-1}$$

By (6.39) and the hypothesis that $\text{supp}(\rho) \in \pi_Y(B(\mathcal{C}_{i,1}, C_2^{-1}\sigma))$, we see that the supports of ρ_i are mutually disjoint, and are disjoint from $\{\pi_Y(\mathcal{C}), \pi_Y(\mathcal{C}_{j,2}), \pi_Y(\mathcal{C}_{j,3})\}_{j=1, \dots, L}$. For each $B = (B_{i,\alpha})_{\substack{1 \leq i \leq L \\ \alpha \in \mathcal{A}}} \in \mathbb{R}^I$, each $x \in X$, we define

$$V(B, x) = \sum_{i=1}^L \sum_{\alpha \in \mathcal{A}} B_{i,\alpha} \rho_i(\pi_Y(x)) V_\alpha(\pi_N(x))$$

It is direct to see that the lift of V , denoted by $G(V)$, is $(Q, C_1 C_2)$ -adapted.

For each $(x, E) \in \mathcal{D}$, there exist indexes $\{\alpha_{i,1}, \dots, \alpha_{i,c_i}\}_{1 \leq i \leq L} \subset \mathcal{A}$ such that for each integer $i \in [1, L]$,

$$(6.42) \quad \det((G(V_{\alpha_{i,k}})(H_{G(F), p^{-1}(\mathcal{C}), p^{-1}(\mathcal{C}_{i,1})}^u(x, E)))_{1 \leq k \leq c_i}) > \kappa$$

Here we identify $H_{F, \mathcal{C}, \mathcal{C}_{i,1}}^u$ with N tacitly.

By (6.41), (6.42) and the fact that the supports of ρ_i are mutually disjoint, we have that

$$D_{B_{j,\alpha_{j,1}}, \dots, B_{j,\alpha_{j,c_j}}} (G(V)(0, H_{G(F), p^{-1}(\mathcal{C}), p^{-1}(\mathcal{C}_{i,1})}^u(x, E), B)) = 0$$

for all $1 \leq i \neq j \leq L$ and

$$\det(D_{B_{i,\alpha_{i,1}}, \dots, B_{i,\alpha_{i,c_i}}} (G(V)(0, H_{G(F), p^{-1}(\mathcal{C}), p^{-1}(\mathcal{C}_{i,1})}^u(x, E), B))) > \kappa$$

for all $1 \leq i \leq L$.

Then we can apply Lemma 1 and Proposition 6 (with $Gr_{c-I}(X)$ in place of X) to conclude the proof.

The case where $F \in \mathcal{U}_2^r(X, Vol)$ follows from a similar argument. \square

6.3. Dichotomy. Combing Proposition 11 with Proposition 9, Proposition 8, we are ready to prove Theorem 2.

Proof of Theorem 2. We first consider the case where $\mathcal{U} = \mathcal{U}_2^r(X, Vol)$. Take any $F \in \mathcal{U}$. Let $\tilde{\chi}_1, \hat{\chi}_1, \tilde{\chi}^c, \hat{\chi}^c$ be given by Definition 11. Let η be given by Proposition 10 with $-\tilde{\chi}^c$ (resp. $\hat{\chi}^c$) in place of χ^s (resp. χ^u). Then by Proposition 11, there exist $L, q_0 > 0$ such that the following is true. Take any compact periodic central leaf \mathcal{C} with periodic $q > q_0$, for any $\epsilon > 0$, there exists $G \in \mathcal{U}_2^r(X, Vol)$ such that $d_{C^r}(G, F) < \epsilon$; and L su -holonomy loops such that the corresponding holonomy maps at \mathcal{C} , denoted by h_1, \dots, h_L , are η -nontransverse to $E_1|_{\mathcal{C}}, E_2|_{\mathcal{C}}$. Moreover, denote $g = G^q|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$, g preserves the splitting $E_1|_{\mathcal{C}} \oplus E_2|_{\mathcal{C}}$. It is direct to verify that $g \in \mathcal{DS}_2^r(\mathcal{C}, m)$ and

$$e^{q\tilde{\chi}^c} < \sup_{v \in E_1 \setminus \{0\}} \frac{\|Dg(v)\|}{\|v\|} < e^{q\tilde{\chi}_1} < e^{q\hat{\chi}_1} < \inf_{u \in E_2 \setminus \{0\}} \frac{\|Dg(u)\|}{\|u\|} < e^{q\hat{\chi}^c}$$

By (3.6), the skew product map F is C^3 and 1-center bunching, hence by Theorem 7 h_1, \dots, h_L are $C^{1+\beta}$ -diffeomorphisms for some $\beta > 0$.

Since in Proposition 10, η only depends on the ratio of the constants related to the map and the splitting. We can apply Proposition 8 and Proposition 10 and for g and $\{h_i\}_{1 \leq i \leq L}$ to conclude that for some integer $K > 0$ the IFS $\{h_i, G^{Kq}h_iG^{-Kq}\}_{1 \leq i \leq L}$ admits no proper closed invariant set of positive measure. We denote that for each $i \in [1, L]$, $G^{Kq}h_iG^{-Kq}$ is also a holonomy map corresponding to a su -loop. Indeed, assume that $h_i = H_{G,\gamma}$ where $\gamma = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ is a su -loop at \mathcal{C} . By the

G -equivariance of G -holonomy maps. We have $G^{Kq}H_{G,\gamma}G^{-Kq} = H_{G,G^{Kq}\gamma}$, here $G^{Kq}\gamma = (G^{Kq}\mathcal{C}_1, G^{Kq}\mathcal{C}_2, G^{Kq}\mathcal{C}_3)$ is a su -loop at \mathcal{C} .

If G has an open accessible class \mathcal{A} . Then by the su -holonomy invariance of accessible classes, the closed set $\mathcal{A}^c \cap \mathcal{C}$ is invariant under all the maps associated to su -holonomy loops, in particular, invariant by the action of $\{h_i, G^{Kq}h_iG^{-Kq}\}$. Then $m(\mathcal{A}^c \cap \mathcal{C}) = 0$. Since for volume preserving skew products, all the holonomy maps preserve m . Hence $(\mu \times m)(\mathcal{A}^c) = 0$. Thus G is essentially accessible.

Finally we prove the C^2 openness. For any \tilde{G} that is sufficiently close to G , denote $\tilde{\mathcal{C}}$ the continuation of \mathcal{C} for \tilde{G} . Moreover, the orbits of $\mathcal{C}, \tilde{\mathcal{C}}$ can be made arbitrarily close by letting G, \tilde{G} to be close in C^1 . For each one of the L su -holonomy loops at \mathcal{C} , denoted by γ , we can find a su -loop for \tilde{G} at $\tilde{\mathcal{C}}$, denoted by $\tilde{\gamma}$, that is close to γ in C^0 . Then by the C^2 closeness between G and \tilde{G} , we can conclude the C^1 -closeness of the holonomy maps $H_{G,\gamma}$ and $H_{\tilde{G},\tilde{\gamma}}$. We note that η -nontransverse condition is a C^1 open condition. This shows that any \tilde{G} in a C^2 neighbourhood of G admits L su -loops at $\tilde{\mathcal{C}}$ which are also η -nontransverse to the dominated splitting of \tilde{G}^q restricted to $\tilde{\mathcal{C}}$. Thus the dichotomy also holds for \tilde{G} .

Using Proposition 9 instead of Proposition 10, *mutatis mutandis* we can extend the proof to the case where $\mathcal{U} = \mathcal{U}_1^r(X, Vol; l), l \in [1, \frac{\epsilon}{2}]$. □

7. THE GENERIC EXISTENCE OF OPEN ACCESSIBLE CLASS

In this section, we will show that within a class of skew products, a generic diffeomorphism admit at least one C^1 -stable open accessible class. Different from the mechanism introduced in [9], where the creation of open accessible class is based on basic homotopy theory, our method is based on a more sophisticated topological lemma discovered in [3].

7.1. A criteria on stable value. In this section we state the topological lemma that is at the core of our construction of open accessible classes. First we borrow a few definitions from [3].

Definition 24. If $f : X \rightarrow Y$ is a continuous map between metric spaces X and Y , then $y \in Y$ is a stable value of f if there is $\epsilon > 0$ such that $y \in \mathfrak{S}(g)$ for every continuous map $g : X \rightarrow Y$ with $d_{C^0}(f, g) < \epsilon$.

Definition 25. A continuous map $f : X \rightarrow Y$ between metric spaces X and Y is called light if all point inverses are totally disconnected.

We will use the following quantified version of Definition 25.

Definition 26. Given constant $\epsilon > 0$, a continuous map $f : X \rightarrow Y$ between metric spaces X and Y is called ϵ -light if for every $y \in Y$, every connected component of $f^{-1}(y)$ has diameter strictly smaller than ϵ .

Now we state the main topological result we will be using in this section.

Theorem 9. For any integer $c > 0$, there exists a constant $\epsilon = \epsilon(c) > 0$ such that any ϵ -light map $f : [0, 1]^c \rightarrow \mathbb{R}^c$ has a stable value.

REMARK 5. In [3], Proposition 3.2, the authors proved that : any light continuous map from a compact metric space of topological dimension at least n to \mathbb{R}^n has stable values. Theorem 9 follows their proof with obvious modifications.

We have the following corollary.

COROLLARY C. For any integer $c \geq 1$, let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open covering of $[0, 1]^c$ such that $\text{diam}(U_\alpha) < \epsilon(c)$ for all $\alpha \in \mathcal{A}$, where $\epsilon(c)$ is given by Theorem 9. Let $f : [0, 1]^c \rightarrow \mathbb{R}^c$ be a continuous map such that for any $x \in [0, 1]^c$, there exist $\mathcal{I} \subset \mathcal{A}$ such that :

- (1) For any $\alpha \in \mathcal{I}$, we have $x \in U_\alpha$;
 - (2) $\bigcap_{\alpha \in \mathcal{I}} f(\partial U_\alpha) = \emptyset$.
- then f has a stable value.

Proof. By Theorem 9, it suffices to check that f is $\epsilon(c)$ -light. Take any $x \in [0, 1]^c$, take $\mathcal{I} \subset \mathcal{A}$ that satisfies (1),(2). Then there exists $\alpha \in \mathcal{I}$ such that $f(x) \notin f(\partial U_\alpha)$. We claim that the connected component of $f^{-1}(f(x))$ containing x is contained in U_α . Indeed, denote the connected component of $f^{-1}(f(x))$ containing x by P . By the continuity of f , $f^{-1}(f(x))$ has no accumulating point in ∂U_α . If $P \cap U_\alpha^c \neq \emptyset$, we can find two disjoint open sets U, V such that $P \subset U \cup V$ and $P \cap U, P \cap V$ are both nonempty. This contradicts the connectedness of P . Hence the claim is true. In particular, the diameter of P is not larger than the diameter of U_α which by hypothesis is strictly smaller than $\epsilon(c)$. Since x is taken to be arbitrary, f is $\epsilon(c)$ -light. \square

7.2. Parametrise an accessible set. In this section, we will parametrise a subset of an accessible class. Then we will construct a family of random perturbations, among which we select the desired diffeomorphism with the help of Corollary C.

Given an integer $c \geq 2$, a positive constant $\theta \in (\frac{c-1}{c}, 1)$, we denote $K_0(c, \theta) = \lceil \frac{c}{c-(c-1)\theta-1} \rceil + 1$ and $K_1(c, \theta) = cK_0(c, \theta) + 1$. In the following, we will fix c, σ and denote $K_1(c, \theta)$ (resp. $K_0(c, \theta)$) as K_1 (resp. K_0)

Now we fix a covering of $[0, 1]^c$ using open sets in \mathbb{R}^2 , denoted by $\{U_\alpha\}_{\alpha \in \mathcal{A}}$. Moreover, we require that :

- (1) \mathcal{A} is a finite set and for all $\alpha \in \mathcal{A}$, there exist constants $\{p_{\alpha,i}, q_{\alpha,i}\}_{i=1, \dots, c} \subset [-1, 2]$ such that $U_\alpha = [p_{\alpha,1}, q_{\alpha,1}] \times \dots \times [p_{\alpha,c}, q_{\alpha,c}]$;
- (2) For any $\alpha \in \mathcal{A}$, $\text{diam}(U_\alpha) < \epsilon(c)$, where $\epsilon(c)$ is given by Theorem 9;
- (3) For each $x \in [0, 1]^c$, there exists a subset $\mathcal{I} \subset \mathcal{A}$ of more than K_1 elements satisfying that $x \in U_\alpha$ for all $\alpha \in \mathcal{I}$, and $\{\partial U_\alpha\}_{\alpha \in \mathcal{I}}$ are mutually disjoint.
- (4) For each integer $i \in [1, c]$, $\{p_{\alpha,i}, q_{\alpha,i}\}_{\alpha \in \mathcal{A}}$ are mutually distinct.

Denote $C_{\min} = 10(\min_{\substack{\alpha \in \mathcal{A} \\ 1 \leq i \leq c}} |p_{\alpha,i} - q_{\alpha,i}|)^{-1}$. We define a set of points in $[0, 6c]$ as follows. For each integer $i \in [1, c]$, we let

$$\mathcal{B}_i = \{p_{\alpha,i}, q_{\alpha,i}\}_{\alpha \in \mathcal{A}}$$

The elements in \mathcal{B}_i are $10C_{\min}^{-1}$ separated for all $i = 1, \dots, c$.

Since $f : Y \rightarrow Y$ is an Anosov map. For any $x, y, z \in Y$ such that $y \in \mathcal{W}_{f,loc}^u(x)$, $z \in \mathcal{W}_{f,loc}^s(x)$, there exist a unique intersection of $\mathcal{W}_{f,loc}^s(y)$ and $\mathcal{W}_{f,loc}^u(z)$, which we denote by $[y, z]$.

LEMMA 11. *There exist a constant $C_0 > 0$ such that for any $y \in Y$, any sufficiently small $\sigma > 0$, there exist $z \in \mathcal{W}_f^s(y, \sigma)$, a curve $\psi : [0, C_0^{-1}\sigma] \rightarrow Y$ contained in $\mathcal{W}_f^u(y)$, parametrised by length with the following properties.*

- (1) *For any $t \in [0, C_0^{-1}\sigma]$, we have $d([\psi(t), z], \mathfrak{S}(\psi)) > C_0^{-1}\sigma$;*
- (2) *For any $s, t \in [0, C_0^{-1}\sigma]$, we have $d(\psi(s), \psi(t)) \geq C_0^{-1}|s - t|$.*

Proof. We will assume that σ is smaller than the injective radius of the manifold Y . Take $C_1 > 1$ and σ small enough such that for any $z \in \mathcal{W}_f^s(y, \sigma)$, any $w \in \mathcal{W}^u(y, \sigma)$, $[w, z]$ is defined and $[w, z] \in B(y, C_1\sigma)$.

Take $C_2 > 0$ such that $d_{\mathcal{W}^u}, d_{\mathcal{W}^s} < C_2 d_Y$, where $d_{\mathcal{W}^u}$ (resp. $d_{\mathcal{W}^s}$) denotes the induced Riemannian metric on \mathcal{W}_f^u (resp. \mathcal{W}_f^s). Since the angles between the stable and unstable directions of the Anosov map $f : Y \rightarrow Y$ is uniformly bounded from below, there exist a constant $C_3 > 0$ such that the following is true when σ is sufficiently small : for any $z \in \mathcal{W}_f^s(y, \sigma) \setminus \mathcal{W}_f^s(y, \frac{1}{2}\sigma)$, $x \in \mathcal{W}_f^u(y, C_3^{-1}\sigma)$, $w \in \mathcal{W}_f^u(z, 2C_1C_2\sigma)$, we have $d(w, x) > 10C_3^{-1}\sigma$.

Now take $C_0 = \max(C_2, C_3)$, take $\psi : [0, C_0^{-1}\sigma]$ to be any geodesic (for $d_{\mathcal{W}^u}$) emanating from y in the unstable leaf, and take z to be an arbitrary point in $\mathcal{W}_f^s(y, \sigma) \setminus \mathcal{W}_f^s(y, \frac{1}{2}\sigma)$. We get (2) by the choice of C_2 and $C_0 \geq C_2$. To see (1), we note that for any $t \in [0, C_0^{-1}\sigma]$, we let $x = \psi(t)$ and $w = [\psi(t), z]$. We have $w \in B(z, \sigma + C_1\sigma) \cap \mathcal{W}_f^u(z)$ by the choice of C_1 , and hence $w \in \mathcal{W}_f^u(z, 2C_1C_2)$ by the choice of C_2 . Finally, we get $d(\psi(t), \mathfrak{S}(\psi)) > C_0^{-1}\sigma$ by $C_0 \geq C_3$. \square

Now we let $C_0 > 0$ be the constant given by Lemma 11. For given $y \in Y$, constant $\sigma > 0$, we define a one-parameter family of 4-legged su -loops as follows. Let $z \in Y$, $\psi : [0, C_0^{-1}\sigma] \rightarrow Y$ be given by Lemma 11, then for each $t \in [0, C_0^{-1}\sigma]$, we define the 4-legged su -loop at N_y associated to parameter t to be

$$\gamma(t) = (N_{\psi(t)}, N_{[\psi(t), z]}, N_z)$$

Since the domain of the curve ψ is $[0, C_0^{-1}\sigma]$, to simplify notations we denote

(1) for each integer $i \in [1, c]$, real number $s \in [-1, 2]$, we denote the normalized coordinate on $[0, C_0^{-1}\sigma]$ as

$$(7.1) \quad \varphi(i, s) = \frac{C_0^{-1}\sigma}{6c}(6i - 2 + s)$$

(2) for each $s = (s_1, \dots, s_c) \in [-1, 2]^c$, each integer $i \in [1, c]$, we define the su -loop corresponding to the i -th coordinate by

$$\gamma_i = \gamma(\varphi(i, s_i))$$

Given $x_0 \in N_y$. For a skew product map $F : N \times Y \rightarrow N \times Y$, we define the following map.

$$(7.2) \quad \begin{aligned} \phi_F : [0, 1]^c &\rightarrow N \\ \phi_F(s_1, \dots, s_c) &= \prod_{i=1}^c H_{F, \gamma_i}(x_0) \end{aligned}$$

Hereafter, for diffeomorphisms f_1, f_2, \dots, f_l , we use the notation $\prod_{i=1}^l f_i$ to denote $f_l \cdots f_1$.

LEMMA 12. *If for all integers $i \in [1, c]$, for any subset $\mathcal{B} \subset \mathcal{B}_i$ containing K_0 elements we have*

$$\bigcap_{r \in \mathcal{B}} \phi_F([-1, 2]^{i-1} \times \{r\} \times [-1, 2]^{c-i}) = \emptyset$$

then ϕ_F has stable values. In particular, F has an open accessible class.

Proof. By Corollary C and the property of the covering $\{U_\alpha\}_\alpha$, it suffices to prove that for any subset $\mathcal{I} \subset \mathcal{A}$ such that $|\mathcal{I}| = K_1$, we have $\bigcap_{\alpha \in \mathcal{I}} \phi_F(\partial U_\alpha) = \emptyset$. Since $\phi_F(\partial U_\alpha) = \bigcup_{i=1}^c \phi_F(\partial_i U_\alpha)$ where

$$\begin{aligned} \partial_i U_\alpha &= [p_{\alpha,1}, q_{\alpha,1}] \times \cdots \times \{p_{\alpha,i}, q_{\alpha,i}\} \times \cdots \times [p_{\alpha,c}, q_{\alpha,c}] \\ &\subset [-1, 2]^{i-1} \times \{p_{\alpha,i}, q_{\alpha,i}\} \times [-1, 2]^{c-i} \end{aligned}$$

we have

$$\bigcap_{\alpha \in \mathcal{I}} \phi_F(\partial U_\alpha) \subset \bigcap_{\alpha \in \mathcal{I}} \left(\bigcup_{i=1}^c \phi_F([-1, 2]^{i-1} \times \{p_{\alpha,i}, q_{\alpha,i}\} \times [-1, 2]^{c-i}) \right)$$

If we have $\bigcap_{\alpha \in \mathcal{I}} \phi_F(\partial U_\alpha) \neq \emptyset$, by $|\mathcal{I}| = K_1 > cK_0$, there exist an integer $i \in [1, c]$, a subset $\mathcal{J} \subset \mathcal{I}$ containing K_0 elements, for each $\beta \in \mathcal{J}$ there is constant $r_\beta \in \{p_{\beta,i}, q_{\beta,i}\}$, such that

$$\bigcap_{\beta \in \mathcal{J}} \phi_F([-1, 2]^{i-1} \times \{r_\beta\} \times [-1, 2]^{c-i}) \neq \emptyset$$

While this contradicts the hypothesis in the lemma since $\{r_\beta\}_{\beta \in \mathcal{J}}$ is a subset of \mathcal{B}_i containing K_0 elements. □

It is also useful to generalise the construction of ϕ_F for a C^r deformation at F as follows.

Given $y \in Y$, $\sigma > 0$. Let $\psi : [0, C_0^{-1}\sigma] \rightarrow Y$ be the curve defined in Lemma 11 and let γ be the one-parameter family of 4-legged su -loops defined as above. Given $x_0 \in N_y$, a skew-product map F , and $\hat{F} : U \times X \rightarrow X$, a C^r deformation with I -parameters at F , we denote

$$\begin{aligned} \Phi : U \times [-1, 2]^c &\rightarrow N_y = N \\ \Phi(b, s_1, \dots, s_c) &= \phi_{\hat{F}(b, \cdot)}(s_1, \dots, s_c) \end{aligned}$$

Similar to (7.2), Φ is related to the holonomy map for T in the following way.

$$(7.3) \quad \Phi(b, s_1, \dots, s_c) = \pi_X\left(\left(\prod_{i=1}^c H_{T, \tilde{\gamma}_i}\right)(b, x_0)\right)$$

As a consequence of Proposition 2 and Theorem 7, we have the following.

LEMMA 13. *If $r > 2$ and $F : X \rightarrow X$ is a 1-center bunching C^r skew product, then after possibly reducing the size of U , for each $s \in [-1, 2]^c$, the map $\Phi(\cdot, s) : U \rightarrow N$ is $C^{1+\beta}$ for some $\beta > 0$.*

By Theorem 5, we have the following.

LEMMA 14. *If $r \geq 1$, $\theta_0 \in (0, 1)$, $F : X \rightarrow X$ is a θ_0 -pinching C^r skew product, then there exist $\theta \in (\theta_0, 1)$, after possibly reducing the size of U , the map $\Phi(b, \cdot) : U \rightarrow N$ is uniformly θ -Holder for all $b \in U$.*

7.3. Creating open accessible class. Now we can perturb F by constructing a sufficiently localized C^r deformation to create an open accessible class. The proof of the following proposition is similar to Proposition 5 and Proposition 11.

PROPOSITION 12. *Let $r > 2$ and F be a 1-center bunching C^r skew product map. Given any non-periodic point $y \in Y$, there exists $\sigma > 0$ such that let γ denote a one-parameter family of 4-legged su -loop associated to y, σ as in Lemma 11, then there exist a constant $\kappa_0 > 0$ and map $\hat{F} : U \times X \rightarrow X$, a volume preserving C^r deformation of I -parameters at F such that the following is true. For any integer $i \in [1, c]$, any $\mathcal{B} \subset \mathcal{B}_i$ containing K_0 elements, for each $r \in \mathcal{B}$ we choose an arbitrary $s_r = (s_{r,1}, \dots, s_{r,c}) \in [-1, 2]^{i-1} \times \{r\} \times [-1, 2]^{c-i}$. Denote for each $B \in T_0 U$,*

$$\Xi(B) = (D\Phi((0, s_{r,1}, \dots, s_{r,c}), B))_{r \in \mathcal{B}}$$

there exists a subspace $H \subset \mathbb{R}^I$ of dimension $K_0 c$ such that we have

$$\det(\Xi|_H) > \kappa_0$$

Proof. There exist a constant $C_1 > 0$ such that for any $\sigma > 0$, any 4-legged su -loop $\gamma(t)$, the part of the local / unstable manifolds connecting the centre leaves in $\gamma(t)$ is contained in $\pi_Y^{-1}(B(y, C_1 \sigma))$. We denote $Q = Q(\sigma) = \pi_Y^{-1}(B(y, C_1 \sigma))$. Since y is non-periodic for f , for any integer $R_0 > 0$, by taking σ to be sufficiently small, we can ensure that $R(Q) > R_0$.

For any $x \in N$, any $v \in T_x N$, there exists a divergence free vector field V_0 such that $V_0(x) = v$. We can choose a finite collection of points in TN , denoted by $\{(x_h, v_h)\}_{h \in \Delta}$, such that the vector fields V_h constructed accordingly satisfy the following property. There exists $\kappa > 0$ such that for each $x \in N$, there exists a set of indexes h_1, \dots, h_c such that

$$\det((V_{h_i}(x))_{1 \leq i \leq c}) > \kappa$$

There exists a constant $C_2 > 0$ such that for any $\sigma > 0$, any $w \in Y$, there exists a smooth function ρ such that

$$\text{supp } \rho \subset B(w, \sigma), \rho(w) = 1 \text{ and } \text{Lip}(\rho) \left\| \sum_{h \in \Delta} V_h \right\|_{C^0} \leq C_2 \sigma^{-1}$$

For each integer $i \in [1, c]$, any $r \in \mathcal{B}_i$, we choose a smooth function $\rho_{r,i}$ such that

$$(7.4) \quad \text{supp } \rho_{r,i} \subset B(\psi(\phi(i, r)), \frac{C_{\min}^{-1} C_0^{-1} \sigma}{6c}), \rho_{r,i}(\psi(\phi(i, r))) = 1$$

and

$$(7.5) \quad \text{Lip}(\rho_{r,i}) \left\| \sum_{h \in \Delta} V_h \right\|_{C^0} \leq 6c C_2 C_{\min} C_0 \sigma^{-1}$$

Denote $I = |\mathcal{A}| |\Delta| c$. For each $B = (B_{r,i,h})_{\substack{r \in \mathcal{B}_i \\ h \in \Delta}} \in \mathbb{R}^I$, we define

$$V(B, x) = \sum_{i=1}^c \sum_{r \in \mathcal{B}_i} \sum_{h \in \Delta} B_{r,i,h} \rho_{r,i}(\pi_Y(x)) V_h(\pi_N(x))$$

By (7.4), (7.1) and fact that the elements in \mathcal{B}_j are $10C_{\min}^{-1}$ separated for each $j = 1, \dots, c$, we see that the support of functions $\rho_{r,i}$ are mutually disjoint. As a consequence, V is $(Q, 6cC_1C_2C_{\min}C_0)$ -adapted.

Differentiate (7.3), we got that for each $s = (s_1, \dots, s_c) \in [-1, 2]^c$

$$(7.6) \quad D\Phi((0, s_1, \dots, s_c), B) \\ = \sum_{l=1}^c D\left(\prod_{j=l+1}^c H_{T, \tilde{\gamma}_j}\right)\left(\left(\prod_{k=1}^l H_{T, \tilde{\gamma}_k}\right)(0, x_0), \cdot\right) \pi_c(DH_{T, \tilde{\gamma}_l}\left(\left(\prod_{k=1}^{l-1} H_{T, \tilde{\gamma}_k}\right)(0, x_0), B\right))$$

Now let $i, \mathcal{B}, \{s_r\}_{r \in \mathcal{B}}$ be given in the lemma. For each $r \in \mathcal{B}$, we denote

$$\gamma_{r,j} = \gamma(\varphi(j, s_{r,j})), \forall 1 \leq j \leq c$$

To simplify notations, for each $r \in \mathcal{B}$, each integer $j \in [1, c]$, we denote

$$\gamma_{r,j} = (N_{y_{r,j,1}}, N_{y_{r,j,2}}, N_{y_{r,j,3}})$$

Now we define H as follows. For each integer $l \in [1, c]$, each $r \in \mathcal{B}$, we denote

$$x_{r,l} = \prod_{k=1}^{l-1} H_{F, \gamma_{r,k}}(x_0)$$

and

$$w_{r,l} = H_{F, y, \psi(\phi(l, s_{r,l}))}^u(x_{r,l}) \in N_{y_{r,l,1}}$$

We choose a set of indexes in Δ , denoted as $h_{r,1}, \dots, h_{r,c} \in \Delta$, such that

$$\det((V_{h_{r,k}}(\pi_N(w_{r,i})))_{1 \leq k \leq c}) > \kappa$$

We denote $E_{r,i,h} = (\delta_{i=i'} \delta_{r=r'} \delta_{h=h'})_{\substack{1 \leq i' \leq c \\ r' \in \mathcal{R} \\ h' \in \Delta}} \in \mathbb{R}^I$ and define

$$H = \oplus_{r \in \mathcal{B}} \oplus_{k=1}^c \mathbb{R} E_{r,i,h_{r,k}}$$

We will estimate $\pi_c(DH_{T, \tilde{\gamma}_{r,l}}((0, x_{r,l}), B))$ for all integers $l \in [1, c]$ and indexes $r \in \mathcal{B}$, then use these estimates to give lower bound for $\det(\Xi|_H)$.

Given integer $l \in [1, c]$, index $r \in \mathcal{B}$, by Proposition 5, we got

$$(7.7) \quad \|\pi_c(DH_{T, \tilde{\gamma}_{r,l}}((0, x_{r,k}), B)) - H_{F, y_{r,l,3}, y}^s H_{F, y_{r,l,2}, y_{r,l,3}}^u \\ H_{F, y_{r,l,1}, y_{r,l,2}}^s(w_{r,l}, V(B, w_{r,l}))\| \lesssim C e^{-R(Q)\xi} \|B\|$$

where $C = 6cC_1C_2C_{\min}C_0$.

For $B \in H$, we have

$$\text{supp} V(B, \cdot) \subset \bigcup_{r \in \mathcal{B}} \text{supp} \rho_{r,i} \times N$$

We also note that $\pi_Y(w_{r,i}) = \psi(\varphi(i, r))$. Then by (7.4), we have $\rho_{r,i}(\pi_Y(w_{r,i})) = 1$. Therefore, we have

$$V(B, w_{r,l}) \\ = \begin{cases} 0, l \neq i \\ \sum_{k=1}^c B_{r,i,h_{r,k}} V_{h_{r,k}}(\pi_N(w_{r,i})), l = i \end{cases}$$

Then

$$(7.8) \quad D_{B_{r',i,h_{r',1}}, \dots, B_{r',i,h_{r',c}}} V(B, w_{r,l}) \\ = \begin{cases} 0, l \neq i \text{ or } r' \neq r \\ (V_{h_{r,k}}(\pi_N(w_{r,i})))_{1 \leq k \leq c}, l = i \text{ and } r' = r \end{cases}$$

Similar to the proof of Lemma 1, by (7.7), (7.6), (7.8) and Proposition 3, we have

$$(7.9) \quad D\Phi((0, s_{r,1}, \dots, s_{r,c}), B) \\ = D\left(\prod_{j=i+1}^c H_{F,\gamma_j}\right) H_{F,y_{r,i,3},y}^s H_{F,y_{r,i,2},y_{r,i,3}}^u H_{F,y_{r,i,1},y_{r,i,2}}^s (w_{r,i}, V(B, w_{r,i})) + O(e^{-R(Q)\xi} \|B\|)$$

Thus by (7.8), we have

$$D_{B_{r',i,h_{r',1}}, \dots, B_{r',i,h_{r',c}}} (D\Phi((0, s_{r,1}, \dots, s_{r,c}), B)) \\ = D\left(\prod_{j=i+1}^c H_{F,\gamma_j}\right) H_{F,y_{r,i,3},y}^s H_{F,y_{r,i,2},y_{r,i,3}}^u H_{F,y_{r,i,1},y_{r,i,2}}^s (w_{r,i}, D_{B_{r',i,h_{r',1}}, \dots, B_{r',i,h_{r',c}}} (V(B, w_{r,i}))) \\ + O(e^{-R(Q)\xi})$$

Hence, when $r \neq r'$, we have

$$\|D_{B_{r',i,h_{r',1}}, \dots, B_{r',i,h_{r',c}}} (D\Phi((0, s_{r,1}, \dots, s_{r,c}), B))\| = O(e^{-R(Q)\xi})$$

when $r = r'$, we have

$$\det D_{B_{r',i,h_{r',1}}, \dots, B_{r',i,h_{r',c}}} (D\Phi((0, s_{r,1}, \dots, s_{r,c}), B)) \gtrsim \kappa - O(e^{-R(Q)\xi})$$

Thus by letting σ to be sufficiently small, we can make $R(Q)$ to be sufficiently large depending on $\kappa, K_0, cC_1C_2C_{\min}C_0$ (all these quantities are indifferent to the choice of σ) and obtain

$$\det(\Xi|_H) \gtrsim \prod_{r \in \mathcal{B}} \frac{1}{2} \kappa > 2^{-K_0} \kappa^{K_0} =: \kappa_0$$

It is clear from the choice of κ that κ_0 is independent of $i, \mathcal{B}, \{s_r\}_{r \in \mathcal{B}}$. This completes the proof. \square

The following proposition is similar to Proposition 6.

PROPOSITION 13. *Assume that F is a $\frac{c-1}{c}$ -pinching, 1-center bunching C^r volume preserving skew product. Given a non-periodic point $y \in Y, \sigma > 0$, a one-parameter family of 4-legged su-loop associated to y, σ as in Lemma 11. If there exist constant $\kappa_0 > 0$ and a C^r volume preserving deformation of I -parameters at F , denoted by \hat{F} , such that for any integer $i \in [1, c]$, any $\mathcal{B} \subset \mathcal{B}_i$ containing K_0 elements, for each $r \in \mathcal{B}$, we choose an arbitrary $s_r = (s_{r,1}, \dots, s_{r,c}) \in [-1, 2]^{i-1} \times \{r\} \times [-1, 2]^{c-i}$, there exists a subspace $H \subset \mathbb{R}^I$ of dimension $K_0 c$ such that we have*

$$\det(H \ni B \mapsto (D\Phi((0, s_r), B))_{r \in \mathcal{B}}) > \kappa_0$$

then for each $\epsilon > 0$, there exists $b \in U$ such that $F' = \hat{F}(b, \cdot)$ satisfies the following.

- (1) $d_{C^r}(F, F') < \epsilon$;
- (2) F' has an open accessible class.

Proof. For each integer $i \in [1, c]$, each subset $\mathcal{B} \subset \mathcal{B}_i$ containing K_0 elements, for each $r \in \mathcal{B}$, we choose an arbitrary $s_r \in [-1, 2]^{i-1} \times \{r\} \times [-1, 2]^{c-i}$, we define map

$$\begin{aligned} \Psi_{\mathcal{B}, \{s_r\}} : U &\rightarrow N^{|\mathcal{B}|} \\ \Psi_{\mathcal{B}, \{s_r\}}(b) &= (\Phi(b, s_r))_{r \in \mathcal{B}} \end{aligned}$$

We denote the diagonal of $N^{|\mathcal{B}|}$ by

$$\Sigma = \{(x)_{r \in \mathcal{B}} | x \in N\}$$

and for any $\delta > 0$, we denote

$$\Sigma_\delta = \{B(x, \delta)^{|\mathcal{B}|} | x \in N\}$$

Then taking the derivative at $0 \in U$, we got

$$D\Psi_{\mathcal{B}, \{s_r\}}(0, B) = (D\Phi((0, s_r), B))_{r \in \mathcal{B}}$$

Then by our hypothesis, there exist $H \subset \mathbb{R}^I$ of dimension $K_0 c$ such that

$$\det(H \ni B \mapsto D\Psi_{\mathcal{B}, \{s_r\}}(0, B)) > \kappa_0$$

Then by Lemma 13, $D\Psi_{\mathcal{B}, \{s_r\}}$ is equi-continuous, there exist $\zeta > 0$, such that for each $b \in U$ and $|b| < \zeta$, there exist $H' \subset T_b U$ such that

$$\dim(H') = K_0 c$$

and

$$(7.10) \quad \det(H' \ni B \mapsto D\Psi_{\mathcal{B}, \{s_r\}}(b, B)) > \frac{1}{2} \kappa_0$$

It is direct to see that ζ can be chosen to be independent of i, \mathcal{B} and $\{s_r\}_{r \in \mathcal{B}}$. Hence for any $b \in U, |b| < \zeta$, any $i \in [1, c]$, any $\mathcal{B} \subset \mathcal{B}_i$ containing K_0 elements, any $s_r \in [-1, 2]^{i-1} \times \{r\} \times [-1, 2]^{c-i}$, there exist a subspace of $T_b U$, denoted by H , of dimension $K_0 c$ such that $\det(T_b U \ni B \mapsto D\Psi_{\mathcal{B}, \{s_r\}}(b, B)) > \frac{1}{2} \kappa_0$.

We choose some sufficiently small $\delta > 0$, some constant $\beta > 0$, for each $i \in [1, c]$, each $r \in \mathcal{B}_i$, choose a $\delta^{\frac{1+\beta}{\theta}}$ -net in $[-1, 2]^{i-1} \times \{r\} \times [-1, 2]^{c-i}$, denoted by $\mathcal{N}_{r,i}$. For each integer $i \in [1, c]$, for each $\mathcal{B} \subset \mathcal{B}_i$ containing K_0 elements, for each $r \in \mathcal{B}$, we choose an arbitrary $s_r = (s_{r,1}, \dots, s_{r,c}) \in \mathcal{N}_{r,i}$, then the lower bound for the determinants implies that

$$\text{Vol}(\Psi_{\mathcal{B}, \{s_r\}}^{-1}(\Sigma_\delta)) \lesssim \kappa_0^{-1} \delta^{cK_0-c}$$

Consider

$$U_1 = \bigcup_{\substack{i \in [1, c] \\ \mathcal{B} \subset \mathcal{B}_i, |\mathcal{B}| = K_0 \\ \{s_r\} \in \prod_{r \in \mathcal{B}} \mathcal{N}_{r,i}}} (\Psi_{\mathcal{B}, \{s_r\}}^{-1}(\Sigma_\delta))$$

then we have

$$\text{Vol}(U_1) \lesssim \kappa_0^{-1} \delta^{-(c-1)K_0 \frac{1+\beta}{\theta}} \delta^{cK_0-c}$$

Since $K_0 > \frac{c}{c-(c-1)\theta^{-1}}$, we have $\frac{\theta c(K_0-1)}{(c-1)K_0} > 1$. We choose $\beta \in (0, \frac{\theta c(K_0-1)}{(c-1)K_0} - 1)$ so that $-(c-1)K_0 \frac{1+\beta}{\theta} + cK_0 - c > 0$. Thus $\text{Vol}(U_1)$ tends to 0 as δ tends to 0. Then

for any $\epsilon > 0$, by letting δ to be sufficiently small, we can find $b \in U \setminus U_1$ so that $d_{C^r}(F, \hat{F}(b, \cdot)) < \epsilon$.

Now we claim that for all sufficiently small $\delta > 0$, any $b \in U \setminus U_1$, take $F' = \hat{F}(b, \cdot)$, the associated map $\phi_{F'} : [0, 1]^c \rightarrow N$ has stable values.

By Lemma 12, it suffices to show that for any integer $i \in [1, c]$, any $\mathcal{B} \subset \mathcal{B}_i$ containing K_0 elements, for any $r \in \mathcal{B}$, we choose an arbitrary $s_r = (s_{r,1}, \dots, s_{r,c}) \in [-1, 2]^{i-1} \times \{r\} \times [-1, 2]^{c-i}$, we have :

$$(7.11) \quad \{\phi_{F'}(s_r)\}_{r \in \mathcal{B}} \text{ do not coincide.}$$

By hypothesis, for each $r \in \mathcal{B}$, there exist $t_r = (t_{r,1}, \dots, t_{r,c}) \in \mathcal{N}_{r,i}$ such that $d(s_r, t_r) < \delta^{\frac{1+\beta}{\theta}}$. Then by the fact that $b \in U \setminus U_1$, we get

$$d(\Psi_{\mathcal{B}, \{t_r\}}(b), \Sigma) > \delta$$

Hence there exists $r, r' \in \mathcal{B}$ such that $d(\Phi(b, t_r), \Phi(b, t_{r'})) > \delta$.

By Lemma 14, $\Phi(b, \cdot) = \phi_{\hat{F}(b, \cdot)}(\cdot) = \phi_{F'}(\cdot)$ is uniformly θ -Holder for $b \in U$, we have

$$d(\phi_{F'}(t_r), \phi_{F'}(s_r)) \lesssim \delta^{1+\beta}$$

Similarly $d(\phi_{F'}(t_{r'}), \phi_{F'}(s_{r'})) \lesssim \delta^{1+\beta}$. Hence when δ can be made arbitrarily small, we have

$$\begin{aligned} d(\phi_{F'}(s_{r'}), \phi_{F'}(s_r)) &> d(\phi_{F'}(t_r), \phi_{F'}(t_{r'})) - d(\phi_{F'}(t_r), \phi_{F'}(s_r)) - d(\phi_{F'}(t_{r'}), \phi_{F'}(s_{r'})) \\ &> \delta - O(\delta^{1+\beta}) > 0 \end{aligned}$$

Hence (7.11). This finishes the proof. \square

Combing Lemma 12 and Proposition 13, we are ready to proof Theorem 1.

Proof of Theorem 1. Take any non-periodic point $y \in Y$. For any $F \in \mathcal{U}$, we first apply Lemma 12 to obtain a volume preserving C^r deformation at F that verify that hypothesis of Proposition 13. Then we apply Proposition 13 to get diffeomorphisms with an open accessible class in arbitrary small C^r neighborhood of F . Since the condition in Lemma 12 is C^0 robust, and the fact that the holonomy maps of C^1 close diffeomorphisms in C^1 are C^0 close, we see that any map that is sufficiently close to F in C^1 topology also has an open accessible class. \square

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